

Measuring and Relating Aggregate and Subaggregate Productivity Change without Neoclassical Assumptions

Bert M. Balk

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Explanation of symbols

.	= data not available
*	= provisional figure
x	= publication prohibited (confidential figure)
—	= nil or less than half of unit concerned
—	= (between two figures) inclusive
0 (0,0)	= less than half of unit concerned
blank	= not applicable
2007–2008	= 2007 to 2008 inclusive
2007/2008	= average of 2007 up to and including 2008
2007/'08	= crop year, financial year, school year etc. beginning in 2007 and ending in 2008
2005/'06–2007/'08	= crop year, financial year, etc. 2005/'06 to 2007/'08 inclusive

Due to rounding, some totals may not correspond with the sum of the separate figures.

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Measuring and Relating Aggregate and Subaggregate Productivity Change without Neoclassical Assumptions

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Abstract

This paper considers the relation between (total factor) productivity measures for individual production units and those for aggregates such as industries, sectors, or economies. This topic has been treated in a number of influential publications, such as Hulten (1978), Gollop (1979), and Jorgenson, Gollop and Fraumeni (1987). What distinguishes this paper from other publications in this area is that I deliberately avoid the making of all kinds of (neoclassical) structural and behavioural assumptions, such as the existence of production frontiers with certain properties, or optimizing behaviour of the production units. In addition, I also treat dynamic ensembles of production units, characterized by entry and exit. Thus, a greater level of generality is obtained from which the earlier results follow as special cases.

Keywords: Producer; profit; profitability; productivity; aggregation; index number theory.

JEL code: C43, O47.

1 Introduction

In a previous paper (Balk 2008a) I considered the measurement of productivity change for a single, consolidated production unit. This paper continues by considering an ensemble of such units, and studying the relation between aggregate and subaggregate (or individual) measures of productivity change. The theory developed here is applicable to a variety of situations, such as 1) a firm consisting of a number of plants, 2) an industry consisting of a number of firms, and 3) an economy or, more precisely, the market sector of an economy consisting of a number of industries.

On an intuitive level the relation between aggregate and subaggregate, or individual, measures of productivity change is not too difficult to understand. Productivity is output quantity divided by input quantity and, thus, productivity change is output quantity change divided by input quantity change. Any aggregate is somehow the sum of its parts, which in the present context implies that aggregate productivity is somehow a weighted mean of subaggregate (or individual) productivities, where the weights somehow express the “importance” of the subaggregates, or individual units, making out

the aggregate. Hence, there are two, independent, factors responsible for aggregate productivity change: 1) productivity change at subaggregate, or individual, level, and 2) change of the “importance” of the subaggregate, or individual, units.

Coming down to practice, things are rapidly becoming complicated. First, firms produce and use multiple commodities, which brings input and output prices into play. Quantities of different commodities cannot be added, but must be aggregated by means of prices. Through time, prices are also changing, which implies that we cannot simply talk about aggregate prices and quantities, but must talk about price and quantity index numbers. Moreover, aggregation rules are not unique anymore. Second, firms and industries deliver to each other, which implies that “simple” addition of production units easily leads to a form of double-counting of outputs and inputs. Third, especially when we are dealing with firms or plants an important fact to take into account is the dynamics of growth, decline, birth, and death of production units.

All this is discussed in the coming sections. Section 2 discusses accounting relations. Sections 3 to 7 discuss the case of a static ensemble of production units. The theory developed in these sections is immediately applicable to the situation of an economy consisting of a number of industries. Sections 8 and 9 proceed with the case of a dynamic ensemble. All the relations developed in sections 3 to 9 are in terms of value-added as output measure. Section 10 discusses the link between value-added based and gross-output based productivity measures. Section 11, finally, points to further work.

At first sight, this paper might look rather intimidating because of its abundant use of mathematical symbols. The purpose here is to be precise, and for precision one needs a fairly detailed notation. Moreover, precision is helpful when it comes to implementation. The mathematics of the paper never goes beyond simple addition, subtraction, multiplication, division, and the handling of weighted means. Basically it is an exercise in accounting. What distinguishes this paper from other articles in this area is that I deliberately avoid the making of all kinds of (neoclassical) structural and behavioural assumptions, such as the existence of production frontiers with certain properties or optimizing behaviour of the production units.

2 Accounting identities

Consider an ensemble \mathcal{K} of consolidated production units. For each unit and period the KLEMS-Y accounting identity in nominal values reads

$$C_{KL(k)}^{kt} + C_{EMS}^{kt} + \Pi^{kt} = R^{kt} \quad (k \in \mathcal{K}), \quad (1)$$

where $C_{KL(k)}^{kt}$ denotes the primary input cost, C_{EMS}^{kt} the intermediate input cost, R^{kt} the revenue¹, and Π^{kt} the profit (defined as remainder). Intermediate input cost (on energy commodities, materials, and business services) and revenue concern generally tradable items. The items in the capital and labour classes, however, are specific for each unit; that explains why there is an additional subscript. Cost and revenue are assumed to be positive. Revenue may be positive, negative, or zero.

The KL-VA accounting identity then reads

$$C_{KL(k)}^{kt} + \Pi^{kt} = R^{kt} - C_{EMS}^{kt} \equiv VA^{kt} \quad (k \in \mathcal{K}), \quad (2)$$

where VA^{kt} denotes value added. In this paper it will always be assumed that $VA^{kt} > 0$.

Adding-up the KLEMS-Y relations over all the units would imply double-counting because of deliveries between units. To see this, it is useful to split intermediate input cost and revenue into two parts, respectively concerning units belonging to the ensemble \mathcal{K} and units belonging to the rest of the world. Thus,

$$C_{EMS}^{kt} = \sum_{k' \in \mathcal{K}, k' \neq k} C_{EMS}^{k'kt} + C_{EMS}^{ekt}, \quad (3)$$

where $C_{EMS}^{k'kt}$ is the cost of the intermediate inputs purchased by unit k from unit k' , and C_{EMS}^{ekt} is the cost of the intermediate inputs purchased by unit k from the world beyond the ensemble \mathcal{K} . Similarly,

$$R^{kt} = \sum_{k' \in \mathcal{K}, k' \neq k} R^{kk't} + R^{ket}, \quad (4)$$

where $R^{kk't}$ is the revenue obtained by unit k from delivering to unit k' , and R^{ket} is the revenue obtained by unit k from delivering to units outside of \mathcal{K} . Adding up the KLEMS-Y relations (1) then delivers

¹It is assumed that with respect to output the units operate on a market.

$$\begin{aligned} \sum_{k \in \mathcal{K}} C_{KL(k)}^{kt} + \sum_{k \in \mathcal{K}} \sum_{k' \in \mathcal{K}, k' \neq k} C_{EMS}^{k'kt} + \sum_{k \in \mathcal{K}} C_{EMS}^{ekt} + \sum_{k \in \mathcal{K}} \Pi^{kt} = \\ \sum_{k \in \mathcal{K}} \sum_{k' \in \mathcal{K}, k' \neq k} R^{kk't} + \sum_{k \in \mathcal{K}} R^{ket}. \end{aligned} \quad (5)$$

If for all the tradable commodities output prices are identical to input prices (which means that there are no tax wedges), then the two intra- \mathcal{K} -trade terms cancel, and the foregoing expression reduces to

$$\sum_{k \in \mathcal{K}} C_{KL(k)}^{kt} + \sum_{k \in \mathcal{K}} C_{EMS}^{ekt} + \sum_{k \in \mathcal{K}} \Pi^{kt} = \sum_{k \in \mathcal{K}} R^{ket}. \quad (6)$$

This is the KLEMS-Y accounting relation for the ensemble \mathcal{K} , considered as a consolidated production unit. The corresponding KL-VA relation is then

$$\sum_{k \in \mathcal{K}} C_{KL(k)}^{kt} + \sum_{k \in \mathcal{K}} \Pi^{kt} = \sum_{k \in \mathcal{K}} R^{ket} - \sum_{k \in \mathcal{K}} C_{EMS}^{ekt}, \quad (7)$$

which can be abbreviated as

$$C_{KL(\mathcal{K})}^{\mathcal{K}t} + \Pi^{\mathcal{K}t} = R^{\mathcal{K}t} - C_{EMS}^{\mathcal{K}t} \equiv VA^{\mathcal{K}t}. \quad (8)$$

One verifies immediately that

$$VA^{\mathcal{K}t} = \sum_{k \in \mathcal{K}} VA^{kt}, \quad (9)$$

which is the reason why the KL-VA production model is the natural starting point for studying the relation between individual and aggregate measures of productivity change.

We first consider a static ensemble, that is, a ensemble of which the composition does not change through time.

3 First decomposition

Aggregate profitability in the KL-VA model is defined as aggregate value added divided by aggregate primary input cost,

$$\frac{VA^{\mathcal{K}t}}{C_{KL(\mathcal{K})}^{\mathcal{K}t}} = \frac{\sum_{k \in \mathcal{K}} VA^{kt}}{\sum_{k \in \mathcal{K}} C_{KL(k)}^{kt}}. \quad (10)$$

Using the logarithmic mean repeatedly, it appears that the logarithm of aggregate profitability is a linear function of the logarithms of all the individual profitabilities,

$$\ln \left(\frac{VA^{\mathcal{K}t}}{C_{KL(\mathcal{K})}^{\mathcal{K}t}} \right) = \sum_{k \in \mathcal{K}} \phi^{kt} \ln \left(\frac{VA^{kt}}{C_{KL(k)}^{kt}} \right), \quad (11)$$

with $\phi^{kt} \equiv L(VA^{kt}, C_{KL(k)}^{kt})/L(VA^{\mathcal{K}t}, C_{KL(\mathcal{K})}^{\mathcal{K}t})$ and $L(\cdot)$ being the logarithmic mean.² Notice that, because $L(a, 1)$ is concave, the coefficients ϕ^{kt} do not necessarily add up to 1.

Aggregate profitability change, going from an earlier period 0 (also called base period) to a later period 1 (also called comparison period), is naturally measured by the ratio of period 1 profitability to period 0 profitability

$$\frac{VA^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}1}}{VA^{\mathcal{K}0}/C_{KL(\mathcal{K})}^{\mathcal{K}0}}. \quad (12)$$

Now, using expression (11), the logarithm of aggregate profitability change can be decomposed symmetrically as

$$\begin{aligned} & \ln \left(\frac{VA^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}1}}{VA^{\mathcal{K}0}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) \\ &= (1/2) \sum_{k \in \mathcal{K}} (\phi^{k1} - \phi^{k0}) (\ln(VA^{k1}/C_{KL(k)}^{k1}) + \ln(VA^{k0}/C_{KL(k)}^{k0})) + \\ & \quad (1/2) \sum_{k \in \mathcal{K}} (\phi^{k1} + \phi^{k0}) (\ln(VA^{k1}/C_{KL(k)}^{k1}) - \ln(VA^{k0}/C_{KL(k)}^{k0})) \\ &= \sum_{k \in \mathcal{K}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\ & \quad \sum_{k \in \mathcal{K}} (1/2) (\phi^{k1} + \phi^{k0}) \left(\ln(VA^{k1}/VA^{k0}) - \ln(C_{KL(k)}^{k1}/C_{KL(k)}^{k0}) \right). \quad (13) \end{aligned}$$

Let there be given price and quantity indices such that for each unit the value-added ratio and the primary-inputs-cost ratio can be decomposed into two components,

²The logarithmic mean is, for any two strictly positive real numbers a and b , defined by $L(a, b) \equiv (a - b)/\ln(a/b)$ if $a \neq b$ and $L(a, a) \equiv a$. It has the following properties: (1) $\min(a, b) \leq L(a, b) \leq \max(a, b)$; (2) $L(a, b)$ is continuous; (3) $L(\lambda a, \lambda b) = \lambda L(a, b)$ ($\lambda > 0$); (4) $L(a, b) = L(b, a)$; (5) $(ab)^{1/2} \leq L(a, b) \leq (a + b)/2$; (6) $L(a, 1)$ is concave.

$$VA^{k1}/VA^{k0} = P_{VA}^k(1,0)Q_{VA}^k(1,0) \quad (14)$$

$$C_{KL(k)}^{k1}/C_{KL(k)}^{k0} = P_{KL(k)}^k(1,0)Q_{KL(k)}^k(1,0). \quad (15)$$

Notice that it is not required that all the functional forms of the price and quantity indices be the same. Substituting this into the last part of expression (13) one obtains

$$\begin{aligned} & \ln \left(\frac{VA^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}1}}{VA^{\mathcal{K}0}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) \\ &= \sum_{k \in \mathcal{K}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\ & \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln(P_{VA}^k(1,0)/P_{KL(k)}^k(1,0)) + \\ & \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1,0), \end{aligned} \quad (16)$$

where individual value-added based (total factor) productivity change is defined as the ratio of the output quantity index number over the input quantity index number, $IPROD_{VA}^k(1,0) \equiv Q_{VA}^k(1,0)/Q_{KL(k)}^k(1,0)$ (see Balk 2008a). Thus, according to expression (16), aggregate profitability change can be decomposed into three parts. The first part can be interpreted as the effect of size change of the individual production units. The size of any individual production unit is thereby measured by the coefficient ϕ^{kt} , which is the relative mean of the unit's value added and primary inputs cost. The weight of each individual size change is the logarithm of mean profitability. The second part gives the aggregate effect of differential price change at the output and input side (which is also called terms-of-trade change). The third part gives the aggregate effect of productivity change at the individual unit level.

Let there also be given price and quantity indices such that the aggregate value-added ratio and the aggregate primary-inputs-cost ratio can be decomposed into two parts,

$$VA^{\mathcal{K}1}/VA^{\mathcal{K}0} = P_{VA}^{\mathcal{K}}(1,0)Q_{VA}^{\mathcal{K}}(1,0) \quad (17)$$

$$C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0} = P_{KL(\mathcal{K})}^{\mathcal{K}}(1,0)Q_{KL(\mathcal{K})}^{\mathcal{K}}(1,0), \quad (18)$$

and let aggregate value-added based productivity change, $IPROD_{VA}^{\mathcal{K}}(1, 0)$, be defined as $Q_{VA}^{\mathcal{K}}(1, 0)/Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)$. Substituting all this into expression (16) and moving the aggregate price index numbers from the left-hand side to the right-hand side delivers an expression for aggregate productivity change:

$$\begin{aligned}
& \ln IPROD_{VA}^{\mathcal{K}}(1, 0) \\
&= \sum_{k \in \mathcal{K}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\
& \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln(P_{VA}^k(1, 0)/P_{KL(k)}^k(1, 0)) + \\
& \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1, 0) - \\
& \quad \ln(P_{VA}^{\mathcal{K}}(1, 0)/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)). \tag{19}
\end{aligned}$$

This can also be written as

$$\begin{aligned}
& \ln IPROD_{VA}^{\mathcal{K}}(1, 0) \\
&= \sum_{k \in \mathcal{K}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\
& \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \left(\ln \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right) - \ln \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) \right) + \\
& \quad \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1, 0) + \\
& \quad \left(\sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) - 1 \right) \ln(P_{VA}^{\mathcal{K}}(1, 0)/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)). \tag{20}
\end{aligned}$$

There are four terms here. The first gives the aggregate effect of size change. The second gives the aggregate effect of differential price change at the output and input side of the production units. The third gives the aggregate effect of unit-specific productivity change. The fourth term concerns, what I call, a concavity discrepancy (which, generally, should be negligible).

Interestingly, if for all units $k \in \mathcal{K}$ and time periods $t = 0, 1$ value added equals primary input cost, $VA^{kt} = C_{KL(k)}^{kt}$ or profit $\Pi^{kt} = 0$, and if there is no differential price change, then only the third term remains,

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1, 0), \quad (21)$$

with $\phi^{kt} = VA^{kt}/VA^{\mathcal{K}t} = C_{KL(k)}^{kt}/C_{KL(\mathcal{K})}^{\mathcal{K}t}$ ($t = 0, 1$), which now add up to 1. Thus, in this case, aggregate productivity change, expressed as a percentage, is a weighted mean of individual productivity changes.

4 Second decomposition

Our second decomposition departs from the following, output-side related, identity:³

$$\ln \left(\frac{VA^{\mathcal{K}1}}{VA^{\mathcal{K}0}} \right) = \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{VA^{k1}}{VA^{k0}} \right), \quad (22)$$

where

$$\psi^k \equiv \frac{L \left(\frac{VA^{k1}}{VA^{\mathcal{K}1}}, \frac{VA^{k0}}{VA^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{K}} L \left(\frac{VA^{k1}}{VA^{\mathcal{K}1}}, \frac{VA^{k0}}{VA^{\mathcal{K}0}} \right)} \quad (k \in \mathcal{K}).$$

Aggregate value-added change, measured as a ratio, is thus equal to a weighted geometric mean of individual value-added changes. Notice that the coefficients ψ^k add up to 1. Each coefficient is the (normalized) mean share of production unit k in aggregate value added.

Now, applying (14) and (17), and moving $P_{VA}^{\mathcal{K}}(1, 0)$ from the left-hand to the right-hand side, we obtain

$$\ln Q_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{P_{VA}^k(1, 0) Q_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right). \quad (23)$$

Subtracting from both sides $\ln Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)$ and applying the definition of aggregate value-added based productivity change delivers

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{P_{VA}^k(1, 0) Q_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0) Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right)$$

³See Balk (2003a) or (2008b; 85-86) for a derivation.

$$= \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{P_{VA}^k(1, 0) Q_{VA}^k(1, 0) Q_{KL(k)}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0) Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0) Q_{KL(k)}^k(1, 0)} \right). \quad (24)$$

Using the definition of individual value-added based productivity change, the last expression can be rewritten as

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{K}} \psi^k \ln IPROD_{VA}^k(1, 0) + \\ &\sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right) + \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{Q_{KL(k)}^k(1, 0)}{Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right). \end{aligned} \quad (25)$$

It is straightforward to check that the definition of ψ^k ($k \in \mathcal{K}$) implies that

$$\sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right) = \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{Q_{VA}^{\mathcal{K}}(1, 0)}{Q_{VA}^k(1, 0)} \right). \quad (26)$$

Substituting this into expression (25) one obtains

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{K}} \psi^k \ln IPROD_{VA}^k(1, 0) + \\ &\sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{Q_{VA}^{\mathcal{K}}(1, 0)}{Q_{VA}^k(1, 0)} \right) + \sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{Q_{KL(k)}^k(1, 0)}{Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right). \end{aligned} \quad (27)$$

Alternatively, using (15) and (18), the last term of expression (25) can be rewritten, which results in

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{K}} \psi^k \ln IPROD_{VA}^k(1, 0) + \\ &\sum_{k \in \mathcal{K}} \psi^k \left(\ln \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right) - \ln \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) \right) + \\ &\sum_{k \in \mathcal{K}} \psi^k \ln \left(\frac{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right). \end{aligned} \quad (28)$$

There are three terms here. The first is a weighted mean of unit-specific productivity changes. The second is the aggregate effect of differential price

change at the output and input side of the production units. The third can be interpreted as the aggregate effect of relative size change, where the size of a unit is measured by its primary-input cost share.

Again, if there is no differential price change then the second term at the right-hand side vanishes. Moreover, if for all units $k \in \mathcal{K}$ and time periods $t = 0, 1$ value added equals primary input cost, $VA^{kt} = C_{KL(k)}^{kt}$, or profit $\Pi^{kt} = 0$, then $C_{KL(k)}^{kt}/C_{KL(\mathcal{K})}^{kt} = VA^{kt}/VA^{\mathcal{K}t}$ ($t = 0, 1$), and by using the definition of the logarithmic mean one easily checks that also the third term vanishes.

Under these two conditions expression (28) reduces to

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \psi^k \ln IPROD_{VA}^k(1, 0). \quad (29)$$

This expression starkly looks like expression (21). In (21) each individual productivity change is weighted by the two-period *arithmetic* mean of relative value added $VA^{kt}/VA^{\mathcal{K}t}$, whereas in (29) each individual productivity change is weighted by the two-period normalized *logarithmic* mean of relative value added. Recall that in the case of an arithmetic mean of shares normalization, to ensure that the weights add up to 1, is unnecessary.

Expressions (21) and (29) are both discrete-time versions of the relation stated in Proposition 3 of Gollop (1979) and there proved using a fair number of neoclassical assumptions. In discrete time there are several ways to measure the continuous-time concept “relative value added”.

The first term at the right-hand side of expression (28) and the third term at the right-hand side of expression (20) correspond to what sometimes is called ‘the bottom-up approach’ to the measurement of aggregate (total- or multi-factor) productivity change. In line with this approach, for example, Van den Bergen *et al.* (2008; 400) rather casually remark that “... MFP change (expressed as a percentage) of the big unit can be expressed as a weighted arithmetic average of MFP change of the small units.”⁴ It now appears that in this approach the remainder terms of expressions (28) and (20) are neglected. Is this an important issue?

An indication can be obtained from *The Netherlands Growth Accounts 2007* published by Statistics Netherlands (2008). Table 3.7 gives the value-

⁴Recall that for index numbers in the neighbourhood of 1 their logarithms approximate percentage changes.

added based MFP change of the aggregate ‘commercial sector’ for 2007 relative to 2006 as 1.6 %. The underlying unrounded figure is 1.6287 %. The same table presents percentage changes for the nine constituting branches. Using the underlying unrounded figures and weighting them with the 2006 nominal value-added shares from Table 3.5 delivers as bottom-up result for the ‘commercial sector’ 1.6494 %. Thus the combined effect of differential price change and size change appears to be - 0.0207 %.

As said, this is only an indication, based on a comparison of but two adjacent years. It is likely that when the time span between the two periods compared becomes larger the effect of the two factors, differential price change and size change, will become larger too. Further indications about the magnitude of this effect can be obtained from Zheng (2005) (compare his Tables 4.1 and 4.4 and see his Figure 5.1) and Harper *et al.* (2008) (see the last two lines of their Table 7).

5 Third decomposition

Our third decomposition departs, rather naturally, from the input-side related identity:

$$\ln \left(\frac{C_{KL(\mathcal{K})}^{\mathcal{K}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) = \sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{k0}} \right), \quad (30)$$

where

$$\omega^k \equiv \frac{L \left(\frac{C_{KL(k)}^{k1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{K}} L \left(\frac{C_{KL(k)}^{k1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)} \quad (k \in \mathcal{K}).$$

Aggregate primary-input cost change is thus equal to a weighted geometric mean of individual primary-input cost changes. Notice that the coefficients ω^k add up to 1. Each coefficient is the (normalized) mean share of production unit k in aggregate primary-input cost.

Applying expressions (15) and (18) and performing steps similar to those in the previous section delivers

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \omega^k \ln IPROD_{VA}^k(1, 0) -$$

$$\sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) - \sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{Q_{VA}^k(1, 0)}{Q_{VA}^{\mathcal{K}}(1, 0)} \right). \quad (31)$$

It is straightforward to check that the definition of ω^k ($k \in \mathcal{K}$) implies that

$$\sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) = \sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)}{Q_{KL(k)}^k(1, 0)} \right). \quad (32)$$

Substituting this into expression (31) one obtains

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{K}} \omega^k \ln IPROD_{VA}^k(1, 0) - \\ &\sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)}{Q_{KL(k)}^k(1, 0)} \right) - \sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{Q_{VA}^k(1, 0)}{Q_{VA}^{\mathcal{K}}(1, 0)} \right). \end{aligned} \quad (33)$$

Alternatively, using (14) and (17) expression (31) can be rewritten as

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{K}} \omega^k \ln IPROD_{VA}^k(1, 0) + \\ &\sum_{k \in \mathcal{K}} \omega^k \left(\ln \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right) - \ln \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) \right) - \\ &\sum_{k \in \mathcal{K}} \omega^k \ln \left(\frac{VA^{k1}/VA^{k0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right). \end{aligned} \quad (34)$$

This expression exhibits the same structure as expression (28). However, notice the negative sign in front of the term measuring relative size change. This makes expression (34) less attractive than (28) for practical purposes.

It is left to the reader to check that, if there is no differential price change and if for each production unit value added equals primary input cost, then expression (34) reduces to (29).

6 Asymmetric decompositions

We now turn to a number of asymmetric decompositions. The first departs from the output-side related identity:

$$\frac{VA^{\mathcal{K}1}}{VA^{\mathcal{K}0}} = \sum_{k \in \mathcal{K}} \frac{VA^{k0}}{VA^{\mathcal{K}0}} \frac{VA^{k1}}{VA^{k0}}. \quad (35)$$

Aggregate value-added change is here written as a weighted arithmetic mean of individual value-added changes.

Applying (14) and (17), moving $P_{VA}^{\mathcal{K}}(1, 0)$ from the left-hand to the right-hand side, dividing both sides by $Q_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)$, and using (15) and (18), respectively, leads to

$$IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \frac{VA^{k0}}{VA^{\mathcal{K}0}} IPROD_{VA}^k(1, 0) \frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right)^{-1} \frac{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}}. \quad (36)$$

If there is no differential price change and if for each production unit value added equals primary input cost, then expression (36) reduces to

$$IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \frac{VA^{k1}}{VA^{\mathcal{K}1}} IPROD_{VA}^k(1, 0), \quad (37)$$

or, to compare with earlier expressions,

$$IPROD_{VA}^{\mathcal{K}}(1, 0) - 1 = \sum_{k \in \mathcal{K}} \frac{VA^{k1}}{VA^{\mathcal{K}1}} (IPROD_{VA}^k(1, 0) - 1). \quad (38)$$

Notice that the individual productivity changes are here weighted with comparison period value-added (= primary-input-cost) shares, not with base period shares.

The second asymmetric decomposition departs from

$$\frac{VA^{\mathcal{K}1}}{VA^{\mathcal{K}0}} = \left(\sum_{k \in \mathcal{K}} \frac{VA^{k1}}{VA^{\mathcal{K}1}} \left(\frac{VA^{k1}}{VA^{k0}} \right)^{-1} \right)^{-1}. \quad (39)$$

Here aggregate value-added change is written as a weighted harmonic mean of individual value-added changes. The same steps as above then lead to

$$\left(IPROD_{VA}^{\mathcal{K}}(1, 0) \right)^{-1} = \quad (40)$$

$$\sum_{k \in \mathcal{K}} \frac{VA^{k1}}{VA^{\mathcal{K}1}} \left(IPROD_{VA}^k(1, 0) \frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right)^{-1} \frac{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)^{-1}.$$

If there is no differential price change and if for each production unit value added equals primary input cost, then expression (40) reduces to

$$IPROD_{VA}^{\mathcal{K}}(1, 0) = \left(\sum_{k \in \mathcal{K}} \frac{VA^{k0}}{VA^{\mathcal{K}0}} \left(IPROD_{VA}^k(1, 0) \right)^{-1} \right)^{-1}. \quad (41)$$

Notice that the individual productivity changes are here weighted with base period value-added (= primary-input-cost) shares.

Returning to the more general expressions (36) and (40), we notice that the first is an arithmetic mean using base period value-added shares whereas the second is a harmonic mean using comparison period value added shares. They resemble Laspeyres and Paasche indices respectively. A symmetric expression for aggregate value-added based productivity change is obtained, in the spirit of Fisher, by taking the geometric mean of the two right-hand sides.

We now consider the input-side related identity:

$$\frac{C_{KL(\mathcal{K})}^{\mathcal{K}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} = \sum_{k \in \mathcal{K}} \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{k0}}. \quad (42)$$

Aggregate primary-input cost change is here written as a weighted arithmetic mean of individual primary-input cost changes. Applying (15) and (18), moving $P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)$ from the left-hand to the right-hand side, dividing both sides by $Q_{VA}^{\mathcal{K}}(1, 0)$, and using (14) and (17), respectively, leads to

$$\begin{aligned} \left(IPROD_{VA}^{\mathcal{K}}(1, 0) \right)^{-1} = & \quad (43) \\ & \sum_{k \in \mathcal{K}} \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \left(IPROD_{VA}^k(1, 0) \right)^{-1} \left(\frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \right)^{-1} \\ & \frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \frac{VA^{k1}/VA^{k0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}}. \end{aligned}$$

If there is no differential price change and if for each production unit value added equals primary input cost, then expression (43) reduces to

$$IPROD_{VA}^{\mathcal{K}}(1, 0) = \left(\sum_{k \in \mathcal{K}} \frac{VA^{k1}}{VA^{\mathcal{K}1}} \left(IPROD_{VA}^k(1, 0) \right)^{-1} \right)^{-1}. \quad (44)$$

Notice that the individual productivity changes are here weighted with comparison period value-added (= primary-input-cost) shares.

The final decomposition departs from

$$\frac{C_{KL(\mathcal{K})}^{\mathcal{K}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} = \left(\sum_{k \in \mathcal{K}} \frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{\mathcal{K}1}} \left(\frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{k0}} \right)^{-1} \right)^{-1}. \quad (45)$$

Aggregate primary-input cost change is here expressed as a weighted harmonic mean of individual primary-input cost changes. The same steps as above then lead to

$$\begin{aligned} IPROD_{VA}^{\mathcal{K}}(1, 0) = & \quad (46) \\ & \sum_{k \in \mathcal{K}} \frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{\mathcal{K}1}} IPROD_{VA}^k(1, 0) \frac{P_{VA}^k(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \left(\frac{P_{KL(k)}^k(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right)^{-1} \\ & \left(\frac{VA^{k1}/VA^{k0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right)^{-1}. \end{aligned}$$

If there is no differential price change and if for each production unit value added equals primary input cost, then expression (46) reduces to

$$IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \frac{VA^{k0}}{VA^{\mathcal{K}0}} IPROD_{VA}^k(1, 0). \quad (47)$$

Notice that the individual productivity changes are here weighted with base period value-added (= primary-input-cost) shares.

The four decompositions presented here – in expressions (36), (40), (43), and (46), respectively – exhibit the same structure. They decompose aggregate productivity change into the contributions of the individual production units. Each of these contributions consists of four components: a weighting coefficient, individual productivity change, differential price change at the output

and input side of the unit, and a measure of relative size change, respectively. A second step is needed for aggregating the three unit-specific components, productivity change, differential price change, and size change, respectively, so that aggregate productivity change can be seen as coming from these three “sources”. There are several methods available for executing this second step, such as the method of Gini (1937), which is a generalization of the Fisher price and quantity indices, or the simpler methods developed by Balk (2003a).

The Gini method leads to rather complicated final expressions for aggregate productivity change. The method advised by Balk (2003a), however, leads to final expressions that at first sight look rather complicated but at second sight appear to reduce to expressions (28) and (34). One example is sufficient to demonstrate this; the remaining cases are left to the reader.

Consider expression (36), and let

$$P_{VA,diff}^k(1,0) \equiv \frac{P_{VA}^k(1,0)}{P_{VA}^{\mathcal{K}}(1,0)} \left(\frac{P_{KL(k)}^k(1,0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1,0)} \right)^{-1} \quad (48)$$

denote unit-specific, value-added based differential price change, and

$$S_{KL}^k(1,0) \equiv \frac{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \quad (49)$$

denote unit-specific primary-input-cost based relative size change. With these definitions expression (36) reads

$$IPROD_{VA}^{\mathcal{K}}(1,0) = \sum_{k \in \mathcal{K}} \frac{VA^{k0}}{VA^{\mathcal{K}0}} IPROD_{VA}^k(1,0) P_{VA,diff}^k(1,0) S_{KL}^k(1,0). \quad (50)$$

Applying then the method advised by Balk (2003a) delivers the following multiplicative decomposition of aggregate productivity change:

$$IPROD_{VA}^{\mathcal{K}}(1,0) = \prod_{k \in \mathcal{K}} \left(IPROD_{VA}^k(1,0) \right)^{\theta^k} \prod_{k \in \mathcal{K}} \left(P_{VA,diff}^k(1,0) \right)^{\theta^k} \prod_{k \in \mathcal{K}} \left(S_{KL}^k(1,0) \right)^{\theta^k}, \quad (51)$$

where the exponents are defined as

$$\theta^k \equiv \frac{L(s_k^1, s_k^0)}{\sum_{k \in \mathcal{K}} L(s_k^1, s_k^0)} \quad (k \in \mathcal{K}) \quad (52)$$

with

$$s_k^t \equiv \frac{VA^{k0} IPROD_{VA}^k(t, 0) P_{VA, diff}^k(t, 0) S_{KL}^k(t, 0)}{\sum_{k \in \mathcal{K}} VA^{k0} IPROD_{VA}^k(t, 0) P_{VA, diff}^k(t, 0) S_{KL}^k(t, 0)} \quad (k \in \mathcal{K}; t = 0, 1). \quad (53)$$

Expression (51) can be written additively as

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) = & \quad (54) \\ & \sum_{k \in \mathcal{K}} \theta^k \ln IPROD_{VA}^k(1, 0) + \sum_{k \in \mathcal{K}} \theta^k \ln P_{VA, diff}^k(1, 0) + \sum_{k \in \mathcal{K}} \theta^k \ln S_{KL}^k(1, 0). \end{aligned}$$

This is the same expression as (28), except that the coefficients θ^k look far more complex than the ψ^k . The ψ^k were defined as (normalized) logarithmic means of base period and comparison period value-added shares, whereas the θ^k were defined as (normalized) logarithmic means of base period and updated base period value-added shares, updated with productivity change, differential price change, and size change. Inserting the various definitions, and using relations (14) and (15), however, confirms that

$$\theta^k = \psi^k \quad (k \in \mathcal{K}). \quad (55)$$

7 The differential price change term

In the case of a static ensemble of production units, the preferred decomposition of aggregate productivity change is given by expression (28). Aggregate productivity change is decomposed according to three “sources”: unit-specific productivity change, differential price change, and relative size change. It is especially the second term that is hard to sell to economists. It is readily acknowledged that productivity change at the level of production units “causes” productivity change at the level of the aggregate, and one is also ready to admit that relative growth or decline of production units contributes to aggregate productivity change. But in which sense can productivity change,

defined as output quantity change divided by input quantity change, be dependent on price change? This section is devoted to an exploration of this topic.

First, careful consideration of the expression defining $P_{VA,diff}^k(1, 0)$ reveals that input or output price development as such does not play a role. It is relative price development, that is, price development of the production unit relative to the aggregate, that matters. If, at the production unit level, there is no dispersion of input and output price developments, then $P_{VA,diff}^k(1, 0) = 1$, and the contribution to aggregate productivity change disappears.

Second, economists usually argue in terms of “levels”, and let aggregate output (input) be the sum of unit-specific output (input). The hidden assumption thereby is that output (input) is homogeneous over the production units. However, even if at the level of individual commodities the price is the same for every buyer/seller, then the “price” of the composite input and output commodity will vary over the production units. Simple addition of “quantities” of input or output, thus, is not admissible. The corresponding “prices” play a role in the aggregation process, and compositional differences in the two periods compared in principle influence the measurement of aggregate quantity change. Thus price change must be taken into account as a separate factor.⁵

Third, it is interesting to consider what happens when for all the production units the same deflators are used, which is a situation very common in empirical work on firm-level data. Thus, instead of expressions (14) and (15), which reflect the true decompositions of the value-added and primary-input-cost ratios, we use

$$VA^{k1}/VA^{k0} = P_{VA}^{k*}(1, 0)Q_{VA}^{k*}(1, 0) \quad (56)$$

$$C_{KL(k)}^{k1}/C_{KL(k)}^{k0} = P_{KL(k)}^{k*}(1, 0)Q_{KL(k)}^{k*}(1, 0), \quad (57)$$

with

$$P_{VA}^{k*}(1, 0) \equiv P_{VA}^{\mathcal{K}}(1, 0) \quad (58)$$

$$Q_{VA}^{k*}(1, 0) \equiv (P_{VA}^k(1, 0)/P_{VA}^{\mathcal{K}}(1, 0))Q_{VA}^k(1, 0) \quad (59)$$

$$P_{KL(k)}^{k*}(1, 0) \equiv P_{KL(k)}^{\mathcal{K}}(1, 0) \quad (60)$$

$$Q_{KL(k)}^{k*}(1, 0) \equiv (P_{KL(k)}^k(1, 0)/P_{KL(k)}^{\mathcal{K}}(1, 0))Q_{KL(k)}^k(1, 0). \quad (61)$$

⁵See also Dufour, Tang and Wang (2008) on the importance of the contribution of differential price change to aggregate productivity change.

Then we obtain, instead of expression (28),

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \psi^k \ln IPROD_{VA}^{k*}(1, 0) + \sum_{k \in \mathcal{K}} \psi^k \ln S_{KL}^k(1, 0), \quad (62)$$

where $IPROD_{VA}^{k*}(1, 0) \equiv Q_{VA}^{k*}(1, 0)/Q_{KL(k)}^{k*}(1, 0)$. There are two terms left, the first of which has the same structure as the first term of (28), whereas the second is identical to the third term of (28). The differential price change term, the second term of (28), has vanished, but the unit-specific productivity change term is now “contaminated” by differential price changes, since

$$IPROD_{VA}^{k*}(1, 0) = IPROD_{VA}^k(1, 0) P_{VA, diff}^k(1, 0) \quad (k \in \mathcal{K}). \quad (63)$$

Substituting the definitions of true productivity change and differential price change, the last expression transforms into

$$IPROD_{VA}^{k*}(1, 0) = \frac{VA^{k1}/VA^{k0}}{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}} \frac{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)}{P_{VA}^{\mathcal{K}}(1, 0)} \quad (k \in \mathcal{K}). \quad (64)$$

This is unit-specific profitability change, deflated by the ratio of aggregate output and input price index numbers.

The unavailability of firm-specific prices or price index numbers means that one cannot calculate firm-specific productivity index numbers. The best one can do is calculate firm-specific profitability ratios deflated by aggregate price index numbers. A recent contribution by Foster, Haltiwanger, and Syverson (2008) sheds light upon the consequences of this shortcoming. They worked with a sample of (almost) single-output establishments producing physically homogeneous goods, so that establishment-level output prices and quantities were available and comparable across establishments. There were no deliveries between such establishments. Thus, in a KLEMS-Y framework they could calculate approximations to deflated profitability index numbers, $IPROD^{k*}(1, 0)$, and productivity index numbers, $IPROD^k(1, 0)$.

Two general lessons are important to keep in mind. The first is that there appeared to be appreciable differences between the distributions of $IPROD^k(1, 0)$ and $IPROD^{k*}(1, 0)$. The second is that one should be very careful with interpreting the index numbers $IPROD^{k*}(1, 0)$ as measures of technological change.

8 The case of a dynamic ensemble

We now consider a dynamic ensemble; that is, an ensemble that changes through time. Thus, wherever necessary, we must add a superscript t to \mathcal{K} . The accounting identities as discussed in section 2 remain valid.

For any two time periods compared, whether adjacent or not, a distinction must be made between continuing, exiting, and entering production units. In particular,

$$\mathcal{K}^0 = \mathcal{C}^{01} \cup \mathcal{X}^0 \quad (65)$$

$$\mathcal{K}^1 = \mathcal{C}^{01} \cup \mathcal{N}^1, \quad (66)$$

where \mathcal{C}^{01} denotes the continuing units (available in both periods), \mathcal{X}^0 the exiting units (available in the base period only), and \mathcal{N}^1 the entering units (available in the comparison period only). Aggregate profitability in period t is measured as

$$\frac{VA^{\mathcal{K}^t}}{C_{KL(\mathcal{K})}^{\mathcal{K}^t}} = \frac{\sum_{k \in \mathcal{K}^t} VA^{kt}}{\sum_{k \in \mathcal{K}^t} C_{KL(k)}^{kt}}, \quad (67)$$

and aggregate profitability change from period 0 to period 1 is measured as the ratio of the two profitabilities. Using expression (11) and the distinction between the three groups of units, aggregate profitability change can be written as

$$\begin{aligned} \ln \left(\frac{VA^{\mathcal{K}^1}}{C_{KL(\mathcal{K})}^{\mathcal{K}^1}} \right) - \ln \left(\frac{VA^{\mathcal{K}^0}}{C_{KL(\mathcal{K})}^{\mathcal{K}^0}} \right) = \\ \sum_{k \in \mathcal{C}^{01}} \phi^{k1} \ln \left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \right) + \sum_{k \in \mathcal{N}^1} \phi^{k1} \ln \left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \right) \\ - \sum_{k \in \mathcal{C}^{01}} \phi^{k0} \ln \left(\frac{VA^{k0}}{C_{KL(k)}^{k0}} \right) - \sum_{k \in \mathcal{X}^0} \phi^{k0} \ln \left(\frac{VA^{k0}}{C_{KL(k)}^{k0}} \right). \end{aligned} \quad (68)$$

The first minus the third term at the right-hand side can be decomposed symmetrically like expression (13). Then, using expressions (14), (15), (17), and (18), we finally get

$$\begin{aligned}
\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = & \quad (69) \\
& \sum_{k \in \mathcal{C}^{01}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\
& \sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) \ln P_{VA, diff}^k(1, 0) + \\
& \sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1, 0) + \\
& \left(\sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) - 1 \right) \ln(P_{VA}^{\mathcal{K}}(1, 0)/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)) + \\
& \sum_{k \in \mathcal{N}^1} \phi^{k1} \ln \left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \right) - \sum_{k \in \mathcal{X}^0} \phi^{k0} \ln \left(\frac{VA^{k0}}{C_{KL(k)}^{k0}} \right).
\end{aligned}$$

As might be expected, if $\mathcal{K}^0 = \mathcal{K}^1 = \mathcal{C}^{01}$, then expression (69) reduces to (20). The last three terms of (69) can be rewritten, using the fact that

$$\sum_{k \in \mathcal{C}^{01}} \phi^{k1} + \sum_{k \in \mathcal{N}^1} \phi^{k1} = \sum_{k \in \mathcal{K}^1} \phi^{k1} \equiv \phi^{\mathcal{K}^1} \quad (70)$$

$$\sum_{k \in \mathcal{C}^{01}} \phi^{k0} + \sum_{k \in \mathcal{X}^0} \phi^{k0} = \sum_{k \in \mathcal{K}^0} \phi^{k0} \equiv \phi^{\mathcal{K}^0}. \quad (71)$$

One then obtains

$$\begin{aligned}
\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = & \quad (72) \\
& \sum_{k \in \mathcal{C}^{01}} (\phi^{k1} - \phi^{k0}) \ln \left(\left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \frac{VA^{k0}}{C_{KL(k)}^{k0}} \right)^{1/2} \right) + \\
& \sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) \ln P_{VA, diff}^k(1, 0) + \\
& \sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1, 0) + \\
& \sum_{k \in \mathcal{N}^1} \phi^{k1} \ln \left(\frac{VA^{k1}}{C_{KL(k)}^{k1}} \left(\frac{P_{VA}^{\mathcal{K}}(1, 0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right)^{-1/2} \right) -
\end{aligned}$$

$$\sum_{k \in \mathcal{X}^0} \phi^{k0} \ln \left(\frac{VA^{k0}}{C_{KL(k)}^{k0}} \left(\frac{P_{VA}^{\mathcal{K}}(1,0)}{P_{KL(\mathcal{K})}^{\mathcal{K}}(1,0)} \right)^{1/2} \right) + \\ \left((1/2)(\phi^{\mathcal{K}1} + \phi^{\mathcal{K}0}) - 1 \right) \ln \left(P_{VA}^{\mathcal{K}}(1,0) / P_{KL(\mathcal{K})}^{\mathcal{K}}(1,0) \right).$$

There are six terms here, four of which are already more or less familiar. The first gives the aggregate effect of size change of the continuing production units. The second gives the aggregate effect of differential price change of the continuing units. The third gives the aggregate effect of productivity change of the continuing units. The fourth gives aggregate profitability of the entering units, where each unit's profitability is *deflated* by the square root of the ratio of aggregate output and input price indices. The fifth gives aggregate profitability of the exiting units, where each unit's profitability is *inflated* by the square root of the ratio of aggregate output and input price indices. Taken together, the fourth and fifth term give the net effect of entering and exiting units on aggregate productivity change. The final term concerns, again, a concavity discrepancy (which should be negligible, in general).

If for all units $k \in \mathcal{K}^t$ and time periods $t = 0, 1$ value added equals primary input cost, $VA^{kt} = C_{KL(k)}^{kt}$, and if there is no differential price change, then expression (72) reduces to

$$\ln IPROD_{VA}^{\mathcal{K}}(1,0) = \sum_{k \in \mathcal{C}^{01}} (1/2)(\phi^{k1} + \phi^{k0}) \ln IPROD_{VA}^k(1,0) \quad (73) \\ - (1/2) \left(\sum_{k \in \mathcal{N}^1} \phi^{k1} + \sum_{k \in \mathcal{X}^0} \phi^{k0} \right) \ln \left(P_{VA}^{\mathcal{K}}(1,0) / P_{KL(\mathcal{K})}^{\mathcal{K}}(1,0) \right).$$

This can be checked by recalling that under the assumptions made the coefficients $\phi^{kt} = VA^{kt} / VA^{\mathcal{K}t}$ add up to 1, so that $\phi^{\mathcal{K}t} = 1$ ($t = 0, 1$). Apart from the last term this expression is the same as (21). Notice, however, that the coefficients ϕ^{kt} ($k \in \mathcal{C}^{01}$) do not add up to 1. Notice further that $\sum_{k \in \mathcal{N}^1} \phi^{k1}$ is the total share of the entering production units in aggregate value added of period 1 whereas $\sum_{k \in \mathcal{X}^0} \phi^{k0}$ is the total share of the exiting production units in aggregate value added of period 0.

9 Dynamic ensemble: more decompositions

In the case of continuing, exiting and entering production units, the value-added identity (9) can be detailed as

$$VA^{\mathcal{K}0} = \sum_{k \in \mathcal{C}^{01}} VA^{k0} + \sum_{k \in \mathcal{X}^0} VA^{k0} \equiv \sum_{k \in \mathcal{C}^{01}} VA^{k0} + VA^{\mathcal{X}0} \quad (74)$$

$$VA^{\mathcal{K}1} = \sum_{k \in \mathcal{C}^{01}} VA^{k1} + \sum_{k \in \mathcal{N}^1} VA^{k1} \equiv \sum_{k \in \mathcal{C}^{01}} VA^{k1} + VA^{\mathcal{N}1}. \quad (75)$$

Thus, depending on the period we are looking at, aggregate value added is the sum of aggregate value added of the continuing units and aggregate value added of the exiting or entering units. Similarly, aggregate primary-input cost is the sum of primary-input cost of the continuing units and primary-input cost of the exiting or entering units:

$$C_{KL(\mathcal{K})}^{\mathcal{K}0} = \sum_{k \in \mathcal{C}^{01}} C_{KL(k)}^{k0} + \sum_{k \in \mathcal{X}^0} C_{KL(k)}^{k0} \equiv \sum_{k \in \mathcal{C}^{01}} C_{KL(k)}^{k0} + C_{KL(\mathcal{X})}^{\mathcal{X}0} \quad (76)$$

$$C_{KL(\mathcal{K})}^{\mathcal{K}1} = \sum_{k \in \mathcal{C}^{01}} C_{KL(k)}^{k1} + \sum_{k \in \mathcal{N}^1} C_{KL(k)}^{k1} \equiv \sum_{k \in \mathcal{C}^{01}} C_{KL(k)}^{k1} + C_{KL(\mathcal{N})}^{\mathcal{N}1}. \quad (77)$$

It is now assumed that the sets \mathcal{X}^0 and \mathcal{N}^1 are both non-empty. Using the value-added identities (74)-(75), we obtain as decomposition of aggregate value-added change

$$\ln \left(\frac{VA^{\mathcal{K}1}}{VA^{\mathcal{K}0}} \right) = \sum_{k \in \mathcal{C}^{01}} \psi^k \ln \left(\frac{VA^{k1}}{VA^{k0}} \right) + \psi^{\mathcal{X}\mathcal{N}} \ln \left(\frac{VA^{\mathcal{N}1}}{VA^{\mathcal{X}0}} \right), \quad (78)$$

where

$$\psi^k \equiv \frac{L \left(\frac{VA^{k1}}{VA^{\mathcal{K}1}}, \frac{VA^{k0}}{VA^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{C}^{01}} L \left(\frac{VA^{k1}}{VA^{\mathcal{K}1}}, \frac{VA^{k0}}{VA^{\mathcal{K}0}} \right) + L \left(\frac{VA^{\mathcal{N}1}}{VA^{\mathcal{K}1}}, \frac{VA^{\mathcal{X}0}}{VA^{\mathcal{K}0}} \right)} \quad (k \in \mathcal{C}^{01})$$

$$\psi^{\mathcal{X}\mathcal{N}} \equiv \frac{L \left(\frac{VA^{\mathcal{N}1}}{VA^{\mathcal{K}1}}, \frac{VA^{\mathcal{X}0}}{VA^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{C}^{01}} L \left(\frac{VA^{k1}}{VA^{\mathcal{K}1}}, \frac{VA^{k0}}{VA^{\mathcal{K}0}} \right) + L \left(\frac{VA^{\mathcal{N}1}}{VA^{\mathcal{K}1}}, \frac{VA^{\mathcal{X}0}}{VA^{\mathcal{K}0}} \right)}.$$

Notice that these coefficients add up to 1; that is, $\sum_{k \in \mathcal{C}^{01}} \psi^k + \psi^{\mathcal{X}\mathcal{N}} = 1$. Each coefficient ψ^k is the (normalized) mean share of continuing production unit

k in aggregate value added, whereas $\psi^{\mathcal{X}\mathcal{N}}$ is the (normalized) mean share of the aggregate of exiting and entering units in aggregate value added.

Put otherwise, to the panel of continuing units we have added an artificial continuing unit for which base period value added is given by $VA^{\mathcal{X}0}$ and comparison period value added by $VA^{\mathcal{N}1}$. Aggregate value-added change can then again be written as weighted geometric mean of individual value-added changes.

Applying then (14) and (17), moving $P_{VA}^{\mathcal{K}}(1, 0)$ from the left-hand to the right-hand side, applying the definitions of aggregate and individual value-added based productivity change, and using (15) and (18), we obtain

$$\begin{aligned}
\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = & \sum_{k \in \mathcal{C}^{01}} \psi^k \ln IPROD_{VA}^k(1, 0) + \\
& \sum_{k \in \mathcal{C}^{01}} \psi^k \ln P_{VA, diff}^k(1, 0) + \\
& \sum_{k \in \mathcal{C}^{01}} \psi^k \ln \left(\frac{C_{KL(k)}^{k1}/C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) + \\
& \psi^{\mathcal{X}\mathcal{N}} \ln \left(\frac{(VA^{\mathcal{N}1}/VA^{\mathcal{X}0})/P_{VA}^{\mathcal{K}}(1, 0)}{(C_{KL(\mathcal{N})}^{\mathcal{N}1}/C_{KL(\mathcal{X})}^{\mathcal{X}0})/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) + \\
& \psi^{\mathcal{X}\mathcal{N}} \ln \left(\frac{C_{KL(\mathcal{N})}^{\mathcal{N}1}/C_{KL(\mathcal{X})}^{\mathcal{X}0}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}/C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right). \tag{79}
\end{aligned}$$

There are five terms here. The first is a weighted sum of unit-specific productivity changes of the continuing units. The second is the aggregate effect of differential price change at the output and input side of the continuing production units. The third can be interpreted as the aggregate effect of relative size change of the continuing units, where the size of a unit is measured by its primary-input cost share.

The remaining two terms are related to the artificial continuing unit. The first of these two terms measures deflated profitability change, deflated by the ratio of aggregate price index numbers. The second measures relative size change, where the size of the artificial unit is measured by its primary-input cost share. Taken together, these two terms measure the net contribution of entering and exiting units to aggregate productivity change.

If for all units $k \in \mathcal{K}^t$ and time periods $t = 0, 1$ value added equals primary

input cost, $VA^{kt} = C_{KL(k)}^{kt}$, and if there is no differential price change, then expression (79) reduces to

$$\begin{aligned} \ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{C}^{01}} \psi^k \ln IPROD_{VA}^k(1, 0) \\ &+ \psi^{\mathcal{KN}} \ln \left(\frac{(VA^{\mathcal{N}1}/VA^{\mathcal{X}0})/P_{VA}^{\mathcal{K}}(1, 0)}{(C_{KL(\mathcal{N})}^{\mathcal{N}1}/C_{KL(\mathcal{X})}^{\mathcal{X}0})/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) \end{aligned} \quad (80)$$

since

$$\sum_{k \in \mathcal{C}^{01}} \psi^k \ln \left(\frac{VA^{k1}/VA^{k0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right) + \psi^{\mathcal{KN}} \ln \left(\frac{VA^{\mathcal{N}1}/VA^{\mathcal{X}0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right) = 0. \quad (81)$$

The first part of expression (80) is the same as (29) except that the sum of the weights $\sum_{k \in \mathcal{C}^{01}} \psi^k \leq 1$. It is also interesting to recall that

$$IPROD_{VA}^k(1, 0) = \frac{Q_{VA}^k(1, 0)}{Q_{KL(k)}^k(1, 0)} = \frac{(VA^{k1}/VA^{k0})/P_{VA}^k(1, 0)}{(C_{KL(k)}^{k1}/C_{KL(k)}^{k0})/P_{KL(k)}^k(1, 0)}, \quad (82)$$

which is production unit k 's deflated profitability change. Thus the two terms at the right-hand side of expression (80) exhibit the same structure.

Using the primary-input cost identities (76)-(77), we obtain as decomposition of primary-input cost change,

$$\ln \left(\frac{C_{KL(\mathcal{K})}^{\mathcal{K}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) = \sum_{k \in \mathcal{C}^{01}} \omega^k \ln \left(\frac{C_{KL(k)}^{k1}}{C_{KL(k)}^{k0}} \right) + \omega^{\mathcal{KN}} \ln \left(\frac{C_{KL(\mathcal{N})}^{\mathcal{N}1}}{C_{KL(\mathcal{X})}^{\mathcal{X}0}} \right), \quad (83)$$

where

$$\begin{aligned} \omega^k &\equiv \frac{L \left(\frac{C_{KL(k)}^{k1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{C}^{01}} L \left(\frac{C_{KL(k)}^{k1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) + L \left(\frac{C_{KL(\mathcal{N})}^{\mathcal{N}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(\mathcal{X})}^{\mathcal{X}0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)} \quad (k \in \mathcal{C}^{01}) \\ \omega^{\mathcal{KN}} &\equiv \frac{L \left(\frac{C_{KL(\mathcal{N})}^{\mathcal{N}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(\mathcal{X})}^{\mathcal{X}0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)}{\sum_{k \in \mathcal{C}^{01}} L \left(\frac{C_{KL(k)}^{k1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(k)}^{k0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right) + L \left(\frac{C_{KL(\mathcal{N})}^{\mathcal{N}1}}{C_{KL(\mathcal{K})}^{\mathcal{K}1}}, \frac{C_{KL(\mathcal{X})}^{\mathcal{X}0}}{C_{KL(\mathcal{K})}^{\mathcal{K}0}} \right)}. \end{aligned}$$

Notice that these coefficients add up to 1; that is, $\sum_{k \in \mathcal{C}^{01}} \omega^k + \omega^{\mathcal{XN}} = 1$. Each coefficient ω^k is the (normalized) mean share of continuing production unit k in aggregate primary-input cost, whereas $\omega^{\mathcal{XN}}$ is the (normalized) mean share of the aggregate of exiting and entering units in aggregate primary-input cost.

Performing the same steps as above we obtain

$$\begin{aligned}
\ln IPROD_{VA}^{\mathcal{K}}(1, 0) &= \sum_{k \in \mathcal{C}^{01}} \omega^k \ln IPROD_{VA}^k(1, 0) + \\
&\sum_{k \in \mathcal{C}^{01}} \omega^k \ln P_{VA, diff}^k(1, 0) - \\
&\sum_{k \in \mathcal{C}^{01}} \omega^k \ln \left(\frac{VA^{k1}/VA^{k0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right) + \\
&\omega^{\mathcal{XN}} \ln \left(\frac{(VA^{\mathcal{N}1}/VA^{\mathcal{X}0})/P_{VA}^{\mathcal{K}}(1, 0)}{(C_{KL(\mathcal{N})}^{\mathcal{N}1}/C_{KL(\mathcal{X})}^{\mathcal{X}0})/P_{KL(\mathcal{K})}^{\mathcal{K}}(1, 0)} \right) - \\
&\omega^{\mathcal{XN}} \ln \left(\frac{VA^{\mathcal{N}1}/VA^{\mathcal{X}0}}{VA^{\mathcal{K}1}/VA^{\mathcal{K}0}} \right). \tag{84}
\end{aligned}$$

There are five terms here. The first is a weighted sum of unit-specific productivity changes of the continuing units. The second is the aggregate effect of differential price change at the output and input side of the continuing production units. The third can be interpreted as the aggregate effect of relative size change of the continuing units, where the size of a unit is measured by its value-added share. Notice the minus sign.

The remaining two terms are related to the artificial continuing unit. The first of these two terms measures deflated profitability change, deflated by the ratio of aggregate price index numbers. The second measures relative size change, where the size of the artificial unit is measured by its value-added share. Notice here also the minus sign. Taken together, these two terms measure the net contribution of entering and exiting units to aggregate productivity change.

If for all units $k \in \mathcal{K}^t$ and time periods $t = 0, 1$ value added equals primary input cost, $VA^{kt} = C_{KL(k)}^{kt}$, and if there is no differential price change, then the same result is obtained as in the previous case.

10 Linking value-added based to gross-output based productivity change

Recall that individual value-added based productivity change, $IPROD_{VA}^k(1, 0)$, is defined as the ratio of an output quantity index number $Q_{VA}^k(1, 0)$ over an input quantity index number $Q_{KL(k)}^k(1, 0)$. In this section we discuss the link between this measure of productivity change and the gross-output based measure, $IPROD^k(1, 0)$, defined as the ratio of an output quantity index number $Q_R^k(1, 0)$ over an input quantity index number $Q_{KL(k)EMS}^k(1, 0)$ (see Balk 2008a).

Balk (2003b) showed that, given price and quantity decompositions of revenue, primary input cost and intermediate input cost, $Q_{KL(k)EMS}^k(1, 0)$ can be defined as the Montgomery-Vartia index of $Q_{KL(k)}^k(1, 0)$ and $Q_{EMS}^k(1, 0)$.⁶ Then

$$\begin{aligned} \ln IPROD^k(1, 0) &= \ln Q_R^k(1, 0) \\ &\quad - \frac{L(C_{KL(k)}^{k1}, C_{KL(k)}^{k0})}{L(C^{k1}, C^{k0})} \ln Q_{KL(k)}^k(1, 0) \\ &\quad - \frac{L(C_{EMS}^{k1}, C_{EMS}^{k0})}{L(C^{k1}, C^{k0})} \ln Q_{EMS}^k(1, 0). \end{aligned} \quad (85)$$

Likewise, $Q_{VA}^k(1, 0)$ can be defined as the Montgomery-Vartia index of $Q_R^k(1, 0)$ and $Q_{EMS}^k(1, 0)$. Then

$$\begin{aligned} \ln IPROD_{VA}^k(1, 0) &= \frac{L(R^{k1}, R^{k0})}{L(VA^{k1}, VA^{k0})} \left[\ln Q_R^k(1, 0) \right. \\ &\quad - \frac{L(VA^{k1}, VA^{k0})}{L(R^{k1}, R^{k0})} \ln Q_{KL(k)}^k(1, 0) \\ &\quad \left. - \frac{L(C_{EMS}^{k1}, C_{EMS}^{k0})}{L(R^{k1}, R^{k0})} \ln Q_{EMS}^k(1, 0) \right]. \end{aligned} \quad (86)$$

Combining these two expressions delivers

⁶Diewert (1978) showed that any superlative index is a second-order approximation to the Montgomery-Vartia index at the point where price and quantity relatives are equal to 1.

$$\begin{aligned} \ln IPROD_{VA}^k(1, 0) = & \frac{L(R^{k1}, R^{k0})}{L(VA^{k1}, VA^{k0})} \left[\ln IPROD^k(1, 0) + \right. \\ & \left(\frac{L(C_{KL(k)}^{k1}, C_{KL(k)}^{k0})}{L(C^{k1}, C^{k0})} - \frac{L(VA^{k1}, VA^{k0})}{L(R^{k1}, R^{k0})} \right) \ln Q_{KL(k)}^k(1, 0) + \\ & \left. \left(\frac{L(C_{EMS}^{k1}, C_{EMS}^{k0})}{L(C^{k1}, C^{k0})} - \frac{L(C_{EMS}^{k1}, C_{EMS}^{k0})}{L(R^{k1}, R^{k0})} \right) \ln Q_{EMS}^k(1, 0) \right]. \end{aligned} \quad (87)$$

The factor in front of the square brackets, $L(R^{k1}, R^{k0})/L(VA^{k1}, VA^{k0})$, is known as the Domar factor: the ratio of (mean) revenue over (mean) value added. One easily checks that if revenue R^{kt} equals cost C^{kt} (or value added VA^{kt} equals primary input cost $C_{KL(k)}^{kt}$, or profit $\Pi^{kt} = 0$) ($t = 0, 1$) then the foregoing expression reduces to

$$\ln IPROD_{VA}^k(1, 0) = \frac{L(R^{k1}, R^{k0})}{L(VA^{k1}, VA^{k0})} \ln IPROD^k(1, 0). \quad (88)$$

This corresponds to Proposition 1 of Gollop (1979). See Jorgenson, Ho and Stiroh (2005; 298) for a modern derivation under the usual neoclassical assumptions.

Return now to the case of a static ensemble and let the equality of revenue and cost (or value added and primary input cost) hold for all the production units $k \in \mathcal{K}$. Assume also that there is no differential price change; that is, $P_{VA, diff}^k(1, 0) = 1$ for all $k \in \mathcal{K}$. Then expression (88) can be substituted into expressions (21) and (29) to yield respectively

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} (1/2)(\phi^{k1} + \phi^{k0}) \frac{L(R^{k1}, R^{k0})}{L(VA^{k1}, VA^{k0})} \ln IPROD^k(1, 0), \quad (89)$$

with $\phi^{kt} = VA^{kt}/VA^{\mathcal{K}t}$ ($k \in \mathcal{K}; t = 0, 1$), and

$$\ln IPROD_{VA}^{\mathcal{K}}(1, 0) = \sum_{k \in \mathcal{K}} \psi^k \frac{L(R^{k1}, R^{k0})}{L(VA^{k1}, VA^{k0})} \ln IPROD^k(1, 0), \quad (90)$$

with $\psi^k = L(\phi^{k1}, \phi^{k0})/\sum_{k \in \mathcal{K}} L(\phi^{k1}, \phi^{k0})$ ($k \in \mathcal{K}$). These two relations connect aggregate value-added based productivity change to disaggregate gross-output based productivity change. Notice that each Domar factor is greater

than or equal to 1, so that the sum of the combined factors preceding $\ln IPROD^k(1, 0)$ is also greater than or equal to 1. Each combined factor approximates (mean) revenue of production unit k over (mean) aggregate value added, and is called a Domar aggregation weight.

The two relations (89) and (90) correspond to relations derived in a continuous-time framework, using a multitude of structural and behavioural assumptions, by Hulten (1978) and Gollop (1979; Proposition 2). See Jorgenson, Ho and Stiroh (2005; 375) for a modern derivation under neoclassical assumptions. The usual interpretation of these relations is that the impact of unit k 's productivity change is greater than corresponds to its share in aggregate production, as measured by value added, because of the intermediate deliveries to other units.

When one does not want to make assumptions about the equality of revenue and cost and the absence of differential price change one must revert to the more intricate expressions which result from the substitution of (87) into (20) or (28) or (34). Interestingly, when (87) is substituted in (27) one obtains the general, discrete-time version of the augmented Domar aggregation formula as derived originally by Jorgenson, Gollop and Fraumeni (1987) under the usual neoclassical assumptions. See Zheng (2005) for a reproduction.

11 Conclusion

In this paper I discussed what Hulten (2001) called the “top-down view of sectoral productivity analysis, in which the aggregate TFP residual is the point of reference.” For linking aggregate value-added based (total factor) productivity change to subaggregate (industrial or individual) productivity change and other sources I recommend, in the case of a static ensemble, the decomposition given by expression (28), and, in the case of a dynamic ensemble, provided that the sets of exiting and entering units are not empty, the extended decomposition given by expression (79). Otherwise, the decomposition given by expression (72) must be used. If one wants to replace subaggregate value-added based productivity change by gross-output based productivity change, and one does not want to make additional assumptions, then expression (87) is the relation to use.

All these relations between aggregate and subaggregate productivity change are mathematical identities. Of course, these relations can be used only when detailed price and quantity data are available. In practice this require-

ment will almost certainly be violated, which necessitates the invocation of various simplifications. The exact relations then provide a starting point for assessing the implications of such simplifications.

What remains to be discussed is the “bottom-up approach [which] takes the universe of plants or firms as the fundamental frame of reference.” Here one freely talks about productivity in terms of levels. Though the main methods linking micro to macro were cursorily reviewed by Balk (2003c)⁷ a reassessment with the instruments developed in this paper is called for.

⁷An additional method was developed by Diewert and Fox (2005).

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