CENTRAL BUREAU OF STATISTICS
The Netherlands
National Accounts Research Division

ON THE ADJUSTMENT OF TABLES WITH LAGRANGE MULTIPLIERS

Rudi Harthoorn
Jan van Dalen

Nr. NA-024
1987

Summary

Tables resulting from estimations for each individual element must often be adjusted. The adjustment with the aid of Lagrange multipliers is in general preferred above the application of the socalled RAS method. In contrast with the public opinion the Lagrange method appears to consume not much more computer time and central memory as the RAS method; for pronounced rectangular tables the Lagrange method appears to be far more economic. This can be obtained by partitioning the resulting matrix equation in such a way that we can get rid of the many zero's. With the Lagrange method it is also possible to adjust the estimations of the row and column sums.

Some special cases of tables to be adjusted are discussed: additional constraints (which is important for input-output tables with estimations for the sum of the values of products which may be substituted for each other), identical corresponding row and column sums, two mutual connected tables (for instance so-called make and use matrices needed for compiling input-output tables), three-dimensional tables and three-dimensional tables with identical corresponding row and column sums (for instance regional input-output tables).

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1. Introduction

Matrices in which the row and column sums are known quite accurately but of which the information about the contents is incomplete or less detailed are a regular occurrence. The consequence is a matrix in which the row and column sums do not correspond with the information about these sums; such a table is inconsistent.

The situation described above can be encountered in many fields. It is very characteristic of sample surveys and surveys which are very sensitive to various possible errors. It can also occur in the compilation of input-output tables or in the construction of input-output tables from a recent version in countries which do not compile them annually. Besides, the 'inconsistency' of input-output tables for a future year on the basis of one scenario or another should not always be considered as a problem[1].

In practice inconsistent tables are usually adjusted to the desired sums with the aid of the popular RAS method. Stone[2] used to apply this method to adjust old input-output tables to the sums of a new year. He justified it by the assumption that technological changes that had occurred could be represented in terms of either columns or rows. For the initial compilation of input-output tables or the correction of survey or statistical errors this reasoning is obviously not valid. In addition, more recent research has proven Stone's original justification quite unsound[3]. The RAS method multiplies the rows and columns by certain factors in an iterative process. Consequently, the relations between the respective elements are not usually interrupted all that drastically within one row or column.

In addition to the RAS method there is the theoretically much more elegant method of adjustment with the aid of Lagrange multipliers. In this method, the rows and columns are multiplied by factors in such a way that the deviation from the original elements is kept to a minimum. Furthermore, the confidence with which the values of the elements are determined can also be taken into account. On the face of it, therefore, it is very surprising that this method is not applied much more widely in practice. The reasons behind preference for the RAS method are: 1. it is easy to programme; 2. it involves less computer time; 3. it takes up less space in the computer's central memory (with the use of Lagrange multipliers, the total available central memory capacity is often on the verge of being exceeded which leads to drastic increases in the amount of computer time needed); 4. in many cases, the results of the two methods turn out to be more or less comparable. Ideally, the method with the Lagrange multipliers should be opted for in the many cases where 2 and 3 are irrelevant; however, it seldom is.

The present paper demonstrates that the calculation time required for Lagrange multipliers can be drastically reduced, so that in nearly all realistic cases, the Lagrange multipliers will require much less central memory capacity and computer time than the RAS method. The reduction is achieved by means of partitioning the system to be solved and subsequently transforming it into a symmetrical system of equations. Some aspects of this paper were in fact already known to some people[4], but it had never got as far as the literature, see e.g.[5,6,7].

This paper also looks into the consequences of certain additional restraints and elaborates the special case of the additional condition met during the compilation of input-output tables: viz. some row sums being identical to corresponding column sums. Possible applications of Lagrange multipliers in higher dimensional tables are also examined with an example of how the method works for three-dimensional tables. The situation of the additional restraint that a certain row sum is identical to the corresponding column sum is also worked out. This situation is characteristic of, among other things, the adjustment of regional input-output tables to overall national tables.

2. Lagrange multipliers

The method is applied in a situation with a matrix with elements a_{ij}, a column vector r with desired row sums r_i and a row vector c' with desired column sums c_j. The following should then apply

$$\sum_{j=1}^{m} a_{kj} = r_k \tag{1}$$

$$\sum_{i=1}^{n} a_{il} = c_l \tag{2}$$

In practice, this requirement is often not fulfilled.

Let us assume that the relative confidences of elements a_{ij} are known; they are represented by g_{ij}. We now look for factors f_{ij} so that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (f_{ij} a_{ij} - a_{ij})^2 / g_{ij} = \text{min.} \tag{3}$$

$$\sum_{i=1}^{n} f_{il} a_{il} = c_l \tag{4}$$

$$\sum_{j=1}^{m} f_{kj} a_{kj} = r_k \tag{5}$$

Equation (3) can also be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (f_{ij} - 1)^2 a_{ij}^2 / g_{ij} = \text{min.} \tag{6}$$

Restraints (4), (5), and (6) can be summed up in the minimization of the Lagrangian L:

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (f_{ij} - 1)^2 a_{ij}^{\ 2} / g_{ij} + \sum_{i=1}^{n} \lambda_i (r_i - \sum_{j=1}^{m} f_{ij} a_{ij}) +$$

$$+ \sum_{j=1}^{m} \mu_j (c_j - \sum_{i=1}^{n} f_{ij} a_{ij}) = \min. \tag{7}$$

The partial derivatives of the Lagrangian with respect to λ_k and μ_l give equations (5) and (6) respectively. The partial derivative with respect to the unknown factor f_{kl} results in

$$\frac{\partial}{\partial f_{kl}} L = (f_{kl} - 1) a_{kl}^{\ 2} / g_{kl} - \lambda_k a_{kl} - \mu_l a_{kl} = 0 \quad (8$$

Therefore

$$(\lambda_k + \mu_l) g_{kl} = (f_{kl} - 1) a_{kl} \tag{9}$$

from which it follows that

$$\sum_{j=1}^{m} (\lambda_k + \mu_j) g_{kj} = \sum_{j=1}^{m} (f_{kj} - 1) a_{kj}$$

$$= \sum_{j=1}^{m} f_{kj} a_{kj} - \sum_{j=1}^{m} a_{kj}$$

$$= r_k - \sum_{j=1}^{m} a_{kj}$$

$$\equiv s_k \tag{10}$$

where s_k is defined as the difference between the existing and the desired sum of the elements of the kth row. Similarly, we find

$$\sum_{i=1}^{n} (\lambda_i + \mu_l) g_{il} = c_l - \sum_{i=1}^{n} a_{il}$$

$$\equiv d_l \tag{11}$$

where d_l is the difference from the desired column sum. Equations (10) and (11) can be combined in matrix notation

$$\begin{bmatrix} s \\ d \end{bmatrix} = \begin{bmatrix} \widehat{G\,i} & G \\ G' & \widehat{i'\,G} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

$$= \begin{bmatrix} \hat{p} & G \\ G' & \hat{q} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \tag{12}$$

where i is a so-called summation vector of the correct length, consisting completely of ones, and the circumflex indicates a diagonal matrix with elments of the vector concerned on the diagonal. The diagonal matrices $\hat{p}$ and $\hat{q}$ comprise the row and column sums respectively of the weight matrix G. The rank of the matrix on the right-hand side of (12) is lower than its order, so that the matrix is singular. This is easy to understand: the diagonal matrices contain information about the compilation of matrix G; even if there lacks one row or column the contents of them can be determined simply from a linear combination of the other rows (or columns), which indicates linear dependency in G. One consequence of this is that there are an infinite number of solutions to (12). We opt for the solution whereby the contribution of the first multiplier in (12) to the other equations in the system of (12) is zero. This can be done by cancelling the first row and column of the matrix in (12) and cancelling the first element of both the vector s and the vector λ. (Another row and column could also be opted for, e.g. the last ones).

The system in (12) is the classic form in which the Lagrange multipliers are sought to be calculated. Once they have been calculated, they are substituted in (9), and the desired 'consistent' matrix can be subsequently determined with the aid of factors f_{ij}.

3. Transformation and partioning

As the rank is equal to the sum of the row and column dimensions of the matrix to be adjusted minus 1, the system of equations is in danger of quickly becoming unmanageably large. Further examination reveals that the matrix contains many zeros and has a number of specific features, which can be used to an advantage in solving the matrix.

First of all we make use of the fact that the weight matrix G consists completely of nonnegative numbers, so that real square roots of its row and column sums in p and q exist. We transform the system in (12) as follows:

$$\begin{bmatrix} \hat{p}^{-\frac{1}{2}} & 0 \\ 0 & \hat{q}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} s \\ d \end{bmatrix} = \begin{bmatrix} \hat{p}^{-\frac{1}{2}} & 0 \\ 0 & \hat{q}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \hat{p} & G \\ G' & \hat{q} \end{bmatrix} \begin{bmatrix} \hat{p}^{-\frac{1}{2}} & 0 \\ 0 & \hat{q}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \hat{p}^{\frac{1}{2}} & 0 \\ 0 & \hat{q}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \tag{13}$$

Here the square root of the matrix applies to a diagonal matrix so that the (reciprocals of) the square roots of the individual diagonal elements have to be taken. Elaboration of (13) results in

$$\begin{aligned} \begin{bmatrix} \hat{p}^{-\frac{1}{2}} s \\ \hat{q}^{-\frac{1}{2}} d \end{bmatrix} &= \begin{bmatrix} I & \hat{p}^{-\frac{1}{2}} G \, \hat{q}^{-\frac{1}{2}} \\ \hat{q}^{-\frac{1}{2}} G' \, \hat{p}^{-\frac{1}{2}} & I \end{bmatrix} \begin{bmatrix} \hat{p}^{\frac{1}{2}} \lambda \\ \hat{q}^{\frac{1}{2}} \mu \end{bmatrix} \\ &= \begin{bmatrix} I & H \\ H' & I \end{bmatrix} \begin{bmatrix} \hat{p}^{\frac{1}{2}} \lambda \\ \hat{q}^{\frac{1}{2}} \mu \end{bmatrix} \end{aligned} \tag{14}$$

where H is defined as

$$H = \hat{p}^{-\frac{1}{2}} G \, \hat{q}^{-\frac{1}{2}} \tag{15}$$

(14) can also be written as

$$\begin{bmatrix} I & H \\ H' & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{p}^{-\frac{1}{2}} s \\ \hat{q}^{-\frac{1}{2}} d \end{bmatrix} = \begin{bmatrix} \hat{p}^{\frac{1}{2}} \lambda \\ \hat{q}^{\frac{1}{2}} \mu \end{bmatrix} \tag{16}$$

As the system in (14) has become independent due to the cancellation of a row and a column, the matrix on the left-hand side of (16) will not generally be singular and its inverse will therefore exist.

The partitioned inverse gives

$$\begin{bmatrix} (I - H H')^{-1} & - (I - H H')^{-1} H \\ - H' (I - H H')^{-1} & I + H' (I - H H')^{-1} H \end{bmatrix} \begin{bmatrix} \hat{p}^{-\frac{1}{2}} s \\ \hat{q}^{-\frac{1}{2}} d \end{bmatrix} =$$

$$= \begin{bmatrix} \hat{p}^{\frac{1}{2}} s \\ \hat{q}^{\frac{1}{2}} d \end{bmatrix} \tag{17}$$

From which follows

$$\begin{aligned} \hat{p}^{\frac{1}{2}} \lambda &= (I - H H')^{-1} \hat{p}^{-\frac{1}{2}} s - (I - H H')^{-1} H \hat{q}^{-\frac{1}{2}} d \\ &= (I - H H')^{-1} (\hat{p}^{-\frac{1}{2}} s - H \hat{q}^{-\frac{1}{2}} d) \end{aligned} \tag{18}$$

$$\begin{aligned} \hat{q}^{\frac{1}{2}} \mu &= - H' (I - H H')^{-1} \hat{p}^{-\frac{1}{2}} s + \{ I + H' (I - H H')^{-1} H \} \hat{q}^{-\frac{1}{2}} d \\ &= \hat{q}^{-\frac{1}{2}} d - H' (I - H H')^{-1} (\hat{p}^{-\frac{1}{2}} s - H \hat{q}^{-\frac{1}{2}} d) \\ &= \hat{q}^{-\frac{1}{2}} d - H' \hat{p}^{\frac{1}{2}} \lambda \end{aligned} \tag{19}$$

Having solved system (17) by partitioning it turns out that multipliers μ can be determined directly from λ. Therefore we only have to solve one partition of the system.

Numerically speaking, the inversion of a matrix is less stable than the solution of a system, while it takes costs more computer time. It is therefore sensible to rewrite (18) as

$$(I - H H') (\hat{p}^{\frac{1}{2}} \lambda) = (\hat{p}^{-\frac{1}{2}} s - H \hat{q}^{-\frac{1}{2}} d) \tag{20}$$

The right-hand side can easily be constructed from the known data; the matrix on the left-hand side is symmetrical and posistive-definite. There are efficient space-saving algorithms for solving symmetrical matrices. Such an algorithm for a positive-definite matrix was found the IMSL[8] library.

4. Numerical aspects

In the RAS method, which is less desirable on theoretical grounds, the required memory space for vectors and matrix for an n x m -matrix is $n \times m + 2n + 2m$. If the Lagrange multipliers in (20) are calculated by means of the Cholesky decomposition[8], the required memory space is $\frac{1}{2} (n - 1)^2 + \frac{1}{2} (n - 1)$, versus $(n + m - 1)^2 + 2 (n + m)$ for the classic solution. Obviously it makes sense to choose $n < m$, by transposing the matrix if necessary. The factor $(n - 1)$ is due to the fact that not all n multipliers have to be calculated.

If we take the matrix in Van der Ploeg's paper[7] as an example, with $n = 40$ and $m = 150$, the required central memory space according to the RAS method, the classic solution and the solution presented here is 6380, 36101 and 780 addresses respectively. The classic solution scores much lower than the RAS method in this area, whereas the method presented here does considerably better.

As far as computer time is concerned, the RAS method involves roughly $4\,k\,n\,m$ elementary calculations, where k is the number of iterations (usually somewhere around 30). The classic solution takes $\sim (n + m)^3 / 3$ calculations and the solution presented here $\sim n^3 / 3$. For the above example, the ratio of the required calculation time is 35 : 100 : 1. The method presented here appears to be considerably cheaper, but this is in fact a greatly distorted impression as the reading in of the data is not taken into account. In the method presented here it may easily be necessary to read in the data twice for large matrices, in additon to reading in the original matrix, and sometimes the weight matrix will have to be transposed, involving two more reading-in and writing-to-disk processes. However, the whole matrix would then no longer fit in the central memory so that in the RAS method part of the matrix will have to be reread in for each iteration, while for the classic solution and the Lagrange-multipliers method the situation will in fact be hopeless. Taking into account the time the computer needs for reading in the data, the new procedure is even better than stated above. Only in the case of (nearly) square matrices which only just fit in the central memory of the modern large computers will the RAS method function better as far as computer time is concerned. For a 450 x 450 -matrix, for example, the ratio of the RAS method to the improved method is 4 : 5.

For a large computer with a central memory of 250 kwords it may be assumed that at least 200 kwords are available for the arrays in a small program. This means that a matrix whose smallest dimension is less than 630 can be processed. When the algorithm is applied on a computer with a virtual memory it should be verified that the available part of the central memory is indeed sufficient; if it is not, the required computer time may increase dramatically as parts of the system to be solved will keep on being read in iteratively from the virtual memory.

5. The weight matrix

The weight matrix can be compiled in various ways. If the confidence intervals for every element of the matrix to be adjusted are known, they can be used to compile the weight matrix. If the matrix to be adjusted in is based on a relatively small unbiased sample, it often turns out that the squares of the elementen, a_{ij}^2, should be taken.

It is not unusual for elements with larger values to be estimated more accurately in the compilation of input-output tables. This varying accurracy could be expressed by chrosing the elements themselves as weights. As this would lead to an adjustment in the wrong direction in the case of negative elements, the absolute values $|a_{ij}|$ should be taken.

Obviously it is also possible to choose all the weights identically. Substituting (18) and (19) in (9), we then get:

$$(\lambda_k + \mu_l) g_{kl} = \frac{1}{n} s_k + \frac{1}{m} d_l - \frac{1}{m\,n} \sum_{i=1}^{m} s_i \tag{21}$$

which can be called an additive adjustment. Here it is still assumed that the row dimension is smaller than the column dimension; if it is the other way around, the sum of the differences from the row sums, s_i, is replaced in (21) by that of the differences from the column sums, d_j. It can be seen directly from (21) what happens if the sum of the colomn sums and row sums is not identical. The matrix is then simply adjusted in in such a way that the sum of the column sums is correct. The last row sum is disregarded.

In contrast with the RAS method, in the Lagrange method the sign of the elements can change. This is easy to recognise in he special case of (21). In practice, with $|a_{ij}|$ or a_{ij}^2, this is very rare, however. If this is unacceptable, the whole procedure can be carried out again when an opposite sign has been established. The weights of the elements concerned should then be set at zero (i.e. dclared completely reliable) and certain values should be chosen for the elements themselves (e.g. they could also be set at zero).

6. Uncertainties in the row and column sums

In formulating the problem, we have assumed that the row and column sums are known with absolute certainty. In practice, this will by no means always be the case. It would be just as justifiable to change the row and column sums. The most desired solution is a simultaneous adjustment of the contents of the matrix and the ultimate row and column sums, and to do this in such a way that the respectiv confidentialiti s are taken into account.

The solution to this problem is completely analogous with the formulation of the problem in chapter 2. The matrix to be adjusted should be concatenated with the row and column sums, while assigning a negative sign to these sums. The new matrix to be adjusted in therefore has dimensions (n + 1) x (m + 1). After the adjusting process, the row and column sums of the original n x m part of the matrix are meant to correspond with the row and column added. This implies that the new objective functions, r and c respectively, are vectors with only zeros.

7. Additional restraints

It is quite usual for extra information about parts of the matrix to be adjusted to be available. Mostly, this information will relate to the sum of some specific elements of the table. (An example of such a situation is found in the construction of an IO table with rough etimations for products which can be substituted for each other). Morrison and Thumann[9] approached this problem by introducing a new set of Lagrange multipliers ν. The resulting system is also suitable for computer-technical simplification.

Suppose that the above-mentioned table A with elements a_{ij} has v additional restraints. These additional restraints state that for v sets U_h with pairs of numbers (i,j), the sum of the elements a_{ij} with $(i,j) \in U_h$ is given by t_h. This implies that restraints (3), (4) and (5) should be extended by the following restraint

$$\sum_{(i,j) \in U_h} f_{ij}\, a_{ij} = t_h \qquad (22)$$

which leads to the following Lagrangian

$$L = \sum_{i=1}^{n} \sum_{j=1}^{m} (f_{ij} - 1)^2 \; a_{ij}^{\;2} \; / \; g_{ij} \; +$$

$$+ \sum_{i=1}^{n} \lambda_i \left(r_i - \sum_{j=1}^{m} f_{ij} \; a_{ij} \right) + \sum_{j=1}^{m} \mu_j \left(c_j - \sum_{i=1}^{n} f_{ij} \; a_{ij} \right) +$$

$$+ \sum_{h=1}^{v} \nu_h \left(t_h - \sum_{(i,j) \in U_h} f_{ij} \; a_{ij} \right) \tag{23}$$

The partial derivative with respect to f_{kl} now gives

$$\frac{\partial L}{\partial f_{kl}} = (f_{kl} - 1) \; a_{kl}^{\;2} \; / \; g_{kl} - \lambda_k \; a_{kl} - \mu_l a_{kl} - \sum_{h=1}^{v} \nu_h \; a_{kl} \; \delta_{kl,h} \tag{24}$$

where $f_{kl,h}$ equals one when (k,l) is an element of the set U_h, and zero in all other cases. The equation now to be solved is

$$\left(\lambda_k + \mu_l + \sum_{h=1}^{v} \nu_h \; \delta_{kl,h} \right) g_{kl} = (f_{kl} - 1) \; a_{kl} \tag{25}$$

Substitution of this in (4), (5) and (22) results in a system of simultaneous equations:

$$\sum_{j=1}^{m} \left(\lambda_k + \mu_j + \sum_{h=1}^{v} \nu_h \; \delta_{kj,h} \right) g_{kj} = s_k \tag{26}$$

$$\sum_{i=1}^{n} \left(\lambda_i + \mu_l + \sum_{h=1}^{v} \nu_h \; \delta_{kj,h} \right) g_{il} = d_l \tag{27}$$

$$\sum_{(i,j) \in U_k} \left(\lambda_i + \mu_j + \sum_{h=1}^{v} \nu_h \, \delta_{ij,h} \right) g_{ij} = t_k - \sum_{(i,j) \in U_k} a_{ij}$$

$$= u_k \tag{28}$$

The above equations can also be expressed in matrix notation. This gives a matrix equation where the matrix M to be inverted is composed of 3 x 3 = 9 submatrices

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{29}$$

The inverse, too, $N = M^{-1}$, can be considerd as being composed of 9 submatrices

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \tag{30}$$

For submatrices N_{ij} we now find the following expressions

$$N_{11} = \Psi$$

$$N_{12} = - \Psi \Phi_{12}$$

$$N_{13} = - \Psi \Phi_{13}$$

$$N_{21} = - \Phi_{21} \Psi$$

$$N_{22} = \Omega + \Phi_{21} \Psi \Phi_{12}$$

$$N_{23} = \Delta_{23} + \Phi_{21} \Psi \Phi_{13}$$

$$N_{31} = - \Phi_{31} \Psi$$

$$N_{32} = \Delta_{32} + \Phi_{31} \Psi \Phi_{12}$$

$$N_{33} = \Delta_{33} + \Phi_{31} \Psi \Phi_{13}$$

$$\Psi = (M_{11} - M_{12} \Phi_{21} - M_{13} \Phi_{31})^{-1} \tag{31}$$

$$\Phi_{12} = M_{12} \Omega + M_{13} \Delta_{32}$$

$$\Phi_{13} = M_{12} \Delta_{23} + M_{13} \Delta_{33}$$

$$\Phi_{21} = \Omega M_{21} + \Delta_{23} M_{31}$$

$$\Phi_{31} = \Delta_{32} M_{21} + \Delta_{33} M_{31}$$

$$\Omega = (M_{22} - M_{23} M_{33}^{-1} M_{32})^{-1}$$

$$\Delta_{23} = - \Omega M_{23} M_{33}^{-1}$$

$$\Delta_{32} = - M_{33}^{-1} M_{32} \Omega$$

$$\Delta_{33} = M_{33}^{-1} + M_{33}^{-1} M_{32} \Omega M_{23} M_{33}^{-1}$$

In the set equations of (31) occur three matrix inverses, viz: Ψ, Ω, and M_{33}^{-1}. The dimensions of the matrices to be inverted depend on the dimensions of the submatrices on the diagonal of (29). The structure of the submatrices to be inverted also depends on the form of these submatrices. Both the form and the dimensions of the submatrices in (29) are determined by equations (26) - (28). Equations (26) and (27) give diagonal submatrices which themselves are diagonal matrices. The diagonal submatrix following from (28) can take on any form, but usually turns out to be a sparse matrix, and often even a diagonal matrix.

The rearrangement of equations (26) - (28) in a suitable order, may yield a matrix equation that is suited for the partitioned solution, as in (31), to be carried out optimally by a computer. When all three diagonal are themselves diagonal matrices, M_{33} will obviously also be a diagonal submatrices matrix, while the matrices Ψ^{-1} and Ω^{-1} can in principle be completely filled. In that case the order of (26) - (28) should be chosen in such a way that the dimensions of M_{33} become maximum, while the order of the other two equations is usually irrelevant. Therefore, if $v > m$ we choose the order (28), (26), (27). This situation will be most common in practice; the order of the last two is based on the consideration that this will link up with the terms derived in the preceding chapters for problems without additional restraints.

There is one exception to the above mentioned order preferences: if the two largest diagonal submatrices both comprise more elements than the central memory of the computer can cope with, the order (26), (27), (28) or (26), (28), (27) may sometimes be preferable. In this situation M_{32} and M_{23} are often sparse matrices, which can mean that Ω^{-1} becomes also a sparse matrix. The inverse of Ω^{-1} can then be determined with the aid of special algorithms for sparse matrices [10], or sometimes even by making use of the specific structure of Ω^{-1}. Which of the two orders is preferable then of depends on the structure of the submatrix Ω^{-1} which can be obtained. When at least one element a_{kl} the matrix to be adjusted is subjected to more than one additional restraint, this will result in a diagonal submatrix that is no longer a diagonal matrix but often still symmetrical. If the number of elements in this submatrix is such that there is enough space in the cenral memory to invert an (non)symmetrical matrix with that number of elements, the order (28), (26), (27) is preferable. If there is not enough space, the same considerations apply as in the case of the diagonal matrices where the orders (26), (27), (28) or (26), (28), (27) are preferable.

In practice, usually $v < m$, while there is enough space in the central memory for the inversion of the $v \times v$ -matrix. Taking the above considerations into account, the system should be solved in the order (28), (26), (27). This leads to the following matrix equation:

$$\begin{bmatrix} Z & X & Y \\ X' & \hat{p} & G \\ Y' & G' & \hat{q} \end{bmatrix} \begin{bmatrix} \nu \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} u \\ s \\ d \end{bmatrix} \tag{32}$$

Here the definition of P, Z and G is the same as in (12). The dimensions of submatrices X and Y are $v \times n$ and $v \times m$ respectively and can be filled completely. In most cases however, they will be sparse. The element (k,h) of X contains the sum of the weights belonging to the kth column of A in as far as pairs of numbers (k,j) are part of the set U_h. Submatrix Y is composed in analogy with X. Submatrix Z contains the sum of the weights of the elements with indices that are part of the subset U_h on the diagonal element (h,h). If no pair (i,j) is part of more than one subset U_h, then Z is a diagonal matrix $\hat{z}$. If some pairs (i,j) do occur in more than one subset, then submatrix Z also contains off-diagonal elements. Off-diagonal elements (k,l) in Z contain the sum of the weights belonging to the pairs of numbers which are part of both subset U_k and U_l.

Just as in (13), we can construct a transformation matrix τ^{-1}

$$\hat{\tau} = \begin{bmatrix} \hat{z}^{\frac{1}{2}} & 0 & 0 \\ 0 & \hat{p}^{\frac{1}{2}} & 0 \\ 0 & 0 & \hat{q}^{\frac{1}{2}} \end{bmatrix} \tag{33}$$

This can be used to transform (32)

$$\hat{\tau}^{-1} \begin{bmatrix} Z & X & Y \\ X' & \hat{p} & G \\ Y' & G' & \hat{q} \end{bmatrix} \hat{\tau}^{-1} \quad \hat{\tau} \begin{bmatrix} \nu \\ \lambda \\ \mu \end{bmatrix} = \hat{\tau}^{-1} \begin{bmatrix} u \\ s \\ d \end{bmatrix} \tag{34}$$

or

$$\begin{bmatrix} J & E & F \\ E' & I & H \\ F' & H' & I \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{35}$$

which can also be written as

$$M \ \xi = \alpha \tag{36}$$

This equation can be solved by inverting matrix M

$$\begin{aligned} \xi &= M^{-1} \ \alpha \\ &= N \ \alpha \end{aligned} \tag{37}$$

The form of N is given in (31). If the submatrices of (35) are substituted for submatrices M_{ij} in (31), we then find for Ψ, Φ, Ω and Δ:

$$\Psi = (I - E \Phi_{21} - H \Phi_{31})^{-1}$$

$$\Phi_{12} = E \Omega + F \Delta_{32}$$

$$\Phi_{13} = E \Delta_{23} + F \Delta_{33}$$

$$\Phi_{21} = \Omega E' + \Delta_{23} F'$$

$$\Phi_{31} = \Delta_{32} E' + \Delta_{33} F'$$

$$\Omega = (I - H H')^{-1} \tag{38}$$

$$\Delta_{23} = - \Omega H$$

$$\Delta_{32} = - H' \Omega$$

$$\Delta_{33} = I + H' \Omega H$$

The terms for Ω and Δ are also found in (18) and (19); these can be put to good use in developing an algorithm. From (37) and (31) the transformed multipliers ξ_1 can now be determined:

$$\xi_1 = \Psi \alpha_1 - \Psi \Phi_{12} \alpha_2 - \Psi \Phi_{13} \alpha_3 \tag{39}$$

Instead of solving system (35) we need only solve the following system for ξ_1

$$\Xi \xi_1 = \beta \tag{40}$$

where

$$\Xi = \Psi^{-1}$$

$$= J - E \, \Phi_{21} - F \, \Phi_{31}$$

$$= J - F F' - (E - F H') (I - H H')^{-1} (E' - H F')$$

$$= J - F F' - V \Omega V' \tag{41}$$

and

$$\beta = \alpha_1 - \Phi_{12} \, \alpha_2 - \Phi_{13} \, \alpha_3$$

$$= \alpha_1 - (E - F H') \Omega \alpha_2 + (E - F H') \Omega H \alpha_3 - F \alpha_3$$

$$= \alpha_1 - V \Omega \alpha_2 + V \Omega H \alpha_3 - F \alpha_3 \tag{42}$$

The multipliers ξ_2 can be determined analogously

$$\xi_2 = - \Phi_{21} \Psi \alpha_1 + (\Omega + \Phi_{21} \Psi \Phi_{12}) \alpha_2 + (\Delta_{23} + \Phi_{21} \Psi \Phi_{13}) \alpha_3$$

$$= - \Phi_{21} (\Psi \alpha_1 - \Psi \Phi_{12} \alpha_2 - \Psi \Phi_{13} \alpha_3) + \Omega \alpha_2 + \Delta_{23} \alpha_3$$

$$= - \Phi_{21} \xi_1 + \Omega \alpha_2 + \Delta_{23} \alpha_3$$

$$= - \Omega (V' \xi_1 - \alpha_2 + H \alpha_3) \tag{43}$$

just as ξ_3

$$
\begin{aligned}
\xi_3 &= -\Phi_{31}\,\Psi\,\alpha_1 + (\Delta_{32} + \Phi_{13}\,\Psi\,\Phi_{12})\,\alpha_2 + (\Delta_{33} + \Phi_{31}\,\Psi\,\Phi_{13})\,\alpha_3 \\
&= -\Phi_{31}\,(\Psi\,\alpha_1 - \Psi\,\Phi_{12}\,\alpha_2 - \Psi\,\Phi_{13}\,\alpha_3) + \Delta_{32}\,\alpha_2 + \Delta_{33}\,\alpha_3 \\
&= -\Phi_{31}\,\xi_1 + \Delta_{32} + \Delta_{33}\,\alpha_3 \\
&= -(\Delta_{32}\,E' + \Delta_{33}\,F')\,\xi_1 + \Delta_{32}\,\alpha_2 + \Delta_{33}\,\alpha_3 \\
&= (H'\,\Omega\,E' - F' - H'\,\Omega\,H\,F')\,\xi_1 - H'\,\Omega\,\alpha_2 + \alpha_3 + H'\,\Omega\,H\,\alpha_3 \\
&= (H'\,\Omega\,V' - F')\,\xi_1 - H'\,\Omega\,\alpha_2 + \alpha_3 + H'\,\Omega\,H\,\alpha_3 \\
&= -F'\,\xi_1 + H'\,\Omega\,(V'\,\xi_1 - \alpha_2 + H\,\alpha_3) + \alpha_3 \\
&= -F'\,\xi_1 - H'\,\xi_2 + \alpha_3
\end{aligned}
\tag{44}
$$

Solving the symmetrical system of equations in (32) (assuming that J is a symmetrical submatrix), which is of the order $(n + m + v - 1)$, now comes down to solving the symmetrical system in (40) which is of the order v. However, in order to determine matrix Ξ in (40), which is of the order v, it is necessary first of all t- determine the symmetrical matrix Ω, which requires a matrix inversion of the order $(n - 1)$. Part of the significance of this partitioning lies in the fact that in many cases the system, including the required inversions, can be solved without the virtual memory room having to be used. This means that a problem which was initially too large to be solved at reasonable computer costs can be reduced to more acceptable proportions.

In practice, it may occur that there are so many additonal restraints v that the order of matrix Ξ is too great to solve the corresponding system within the central memory. As mentioned above, either (26), (27), (28) or (26), (28), (27) should be preferred to the case worked out above for (28), (26), (27).

For the first alternative, (26), (27), (28), instead of (35), a permuted form of it has to be solved

$$\begin{bmatrix} I & H & E' \\ H' & I & F' \\ E & F & J \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_1 \end{bmatrix} \tag{45}$$

The solution to this system obviously leads to the same results as in (40) - (44), this solution can be written in another form, however:

$$\Xi^a \ \xi_2 = \beta^a \tag{46}$$

with

$$\Xi^a = I - E' \ J^{-1} \ E - V^a \ \Omega^a \ V^{a\prime} \tag{47}$$

and

$$\beta^a = \alpha_2 - V^a \ \Omega^a \ \alpha_3 + V^a \ \Omega^a \ F' \ J^{-1} \ \alpha_1 - E' \ J^{-1} \ \alpha_1 \tag{48}$$

where

$$\Omega^a = (I - F' \ J^{-1} \ F)^{-1} \tag{49}$$

and

$$V^a = H - E' \ J^{-1} \ F \tag{50}$$

Having solved (46), ξ_1 and ξ_3 can be computed

$$\xi_3 = - \Omega^a (V^{a\prime} \ \xi_2 - \alpha_3 + F' \ J^{-1} \ \alpha_1) \tag{51}$$

$$\xi_1 = - J^{-1} (E \ \xi_2 + F \ \xi_3 - \alpha_1) \tag{52}$$

The second alternative, (26), (28), (27), gives the following solution

$$\begin{bmatrix} I & E' & H \\ E & J & F \\ H' & F' & I \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_1 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_3 \end{bmatrix} \tag{53}$$

which can also be written as

$$\Xi^b \xi_2 = \beta^b \tag{54}$$

with

$$\Xi^b = I - H H' - V^b \Omega^b V^{b\prime} \tag{55}$$

$$\beta^b = \alpha_2 - V^b \Omega^b \alpha_1 + V^b \Omega^b F \alpha_3 - H \alpha_3 \tag{56}$$

$$\Omega^b = (J - F F')^{-1} \tag{57}$$

$$V^b = E' H F' \tag{58}$$

And ξ_1 and ξ_3 can subsequently be computed

$$\xi_1 = - \Omega^b (V^{b\prime} \xi_2 - \alpha_1 + F \alpha_3) \tag{59}$$

$$\xi_3 = - H' \xi_2 - F' \xi_1 \alpha_3 \tag{60}$$

In the first alternative order we are confronted with the calculation of inverses J^{-1} and Ω^a, in the second only with Ω^b. These matrices are usually symmetrical and often sparse. It is sometimes possible to calculate the inverses directly from the special form. Otherwise the algorithms for inversion specially aimed at sparse matrices can be used. The NAG library[11], for example, includes Paige and Saunders' algorithm, which makes use of the so-called Lanczas process.

8. Identical row and column sums

In compiling some tables, input-output tables for example, special situations sometimes arise: in addition to estimates about the contents, (independent) estimates are available for the column and row sums, and furthermore a number of column sums are required to be identical to row sums with same number. This is a special case of the additional restraints described in the previous chapter.

These assumed additional restraints can be described by

$$f_{hm}\, a_{hm} = f_{nh}\, a_{nh} \qquad h \le v < n \le m \tag{61}$$

In this case, this is the analogue of (22). The row and column sums are thus put in the nth row and the mth column with a negative sign (see chapter 6). In consequence, vectors r and c consist entirely of zeros. This leads to the following Lagrangian

$$\begin{aligned} L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (f_{ij} - 1)^2\, a_{ij}{}^2 / g_{ij} \; + \\ - \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} f_{ij}\, a_{ij} - \sum_{j=1}^{m} \mu_j \sum_{i=1}^{n} f_{ij}\, a_{ij} \; + \\ - \sum_{h=1}^{v} \nu_h (f_{hm}\, a_{hm} - f_{nh}\, a_{nh}) = \text{min.} \end{aligned} \tag{62}$$

For the partial derivatives we find

$$\frac{\partial L}{\partial f_{kl}} = (f_{kl} - 1) a_{kl}{}^{2} / g_{kl} - \lambda_k a_{kl} - \mu_l a_{kl} +$$

$$- \nu_k a_{kl} \delta_{lm} \eta_{kv} + \nu_l a_{kl} \delta_{kn} \eta_{lv} \tag{63}$$

where δ_{ij} is the well-known Kronecker delta and η_{ij} is somewhat similar to Heaviside's step function: $i \le j \Rightarrow \eta_{ij} = 1$; $i > j \Rightarrow \eta_{ij} = 0$. Having set the partial derivatives equal to zero, we get the following equation

$$\{ \lambda_k + \mu_l + (\nu_k \delta_{lm} \eta_{kv} - \nu_l \delta_{kn} \eta_{lv}) \} g_{kl} = (f_{kl} - 1) a_{kl} \tag{64}$$

Substituting (62) in (4), (5) and (58) gives

$$\sum_{j=1}^{n} (\lambda_k + \mu_j + \delta_{jm} \nu_k \eta_{kv} - \delta_{kn} \nu_j \eta_{jv}) g_{kj} = s_k \tag{65a}$$

$$\sum_{i=1}^{n} (\lambda_i + \mu_l + \delta_{ln} \nu_i \eta_{iv} - \delta_{in} \nu_l \eta_{lv}) g_{il} = d_l \tag{65b}$$

$$\lambda_h g_{hm} + \mu_m g_{hm} + \nu_h g_{hm} - \lambda_n g_{nh} - \mu_h g_{nh} - \nu_h g_{nh} = a_{nh} - a_{hm} \tag{65c}$$

Equations (65a), (65b) and (65c) are the analogues of (26), (27) and (28) respectively. In (65), the Kronecker deltas with index h are always zero since $h < n \le m$; for the same reason the η-functions with index h are always one.

With the aid of (65a), submatrix X can be inserted in (32):

$$X = \{ x_{ij} \mid x_{ij} = \begin{cases} g_{im}, & i = j \ \wedge \ i \le v \\ - g_{nj}, & i = n \\ 0, & \text{all other } (i,j) \ \} \end{cases} \tag{66a}$$

and (65b) gives the composition of submatrix Y:

$$
\begin{aligned}
Y = \{\, y_{ij} \mid y_{ij} &= -\, g_{ni}, && i = j \quad \wedge \quad i \le v \\
&= g_{jm}, && j = m \\
&= 0, && \text{all other } (i,j) \,\}
\end{aligned}
\tag{66b}
$$

Here, matrix Z in (32) is reduced to a diagonal matrix $\hat{z}$; the elements of z are given by

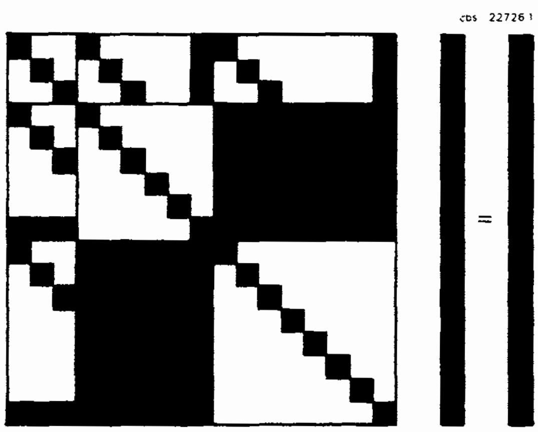
$$z_i = g_{im} + g_{ni} \tag{67}$$

and subvector u is given by

$$u_i = a_{im} - a_{ni} \tag{68}$$

Figure 1 shows where the nonzero elements in equation (32), and therefore also in (35), can be found.

1. **The pattern of equation $M\xi = \alpha$**
n-1=6, m=8, v=3



In chapter 7 we switched to solving the smaller system (40) instead of the large system (35). Matrix Ξ in (40) can be determined with the aid of

(41); but Ω must be known. In this case

$$\Omega = (I - H H')^{-1} \tag{69}$$

where Ω is therefore a symmetrical matrix. The order of Ω is greater than that of Ξ but considerable smaller than that of M. Here, the order of Ω is therefore more or less normative for the required computer capacity.

9. Two different matrices with identical corresponding row and column sums

In the compilation of input-output tables according to the SNA guidelines [12], two matrices, the make and use matrices, are compiled. These two matrices bear some relation to each other: 1. some of the corresponding row sums are identical; 2. some of the corresponding column sums are identical; 3. the total-general is identical for both matrices. If these two matrices are considered as submatrices of a super matrix, the situation can be considered as a special case of the additional restraints in chapter 7.

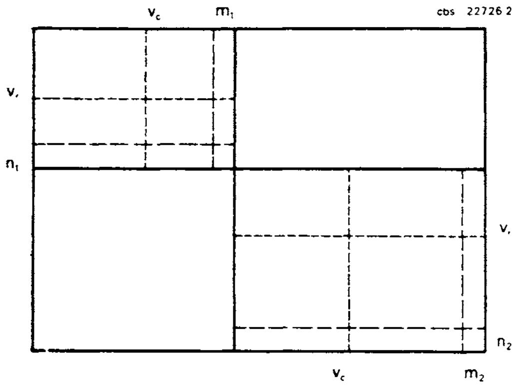
We have an $n_1 \times m_1$ -matrix A_1 with elemants a_{ij1}, of which elements $a_{n_1 j1}$ and $a_{im_1 1}$ are the negative values of the column and row sums respectively of the other elements. Analogously, we also have an $n_2 \times m_2$ -matrix A_2 with elements a_{ij2}. These matrices must have the first v_r row sums and the first v_c column sums in common. The additional restraints can therefore be formulated as

$$\begin{aligned}
f_{k\,m_1 1}\, a_{k\,m_1 1} &= f_{k\,m_2 2}\, a_{k\,m_2 2} && k \le v_r \\
f_{n_1 1\,1}\, a_{n_1 1\,1} &= f_{n_2 1\,2}\, a_{n_2 1\,2} && 1 \le v_c \\
f_{n_1 m_1 1}\, a_{n_1 m_1 1} &= f_{n_2 m_2 2}\, a_{n_2 m_2 2}
\end{aligned} \tag{70}$$

Figure 2 gives a schematic illustration of the composition of the super matrix. The Lagrangian for this case can be compiled from expressions (3), (49), (59) and (70):

2. Schematical representation of the supermatrix

$n_1=6,\ m_1=9,\ n_2=8,\ m_2=11,\ v_r=3,\ v_c=5$



$$
\begin{aligned}
L = & \sum_{i=1}^{n_1} \sum_{j=1}^{m_1} (f_{ij1} - 1)^2 \, a_{ij1}^{\,2} / g_{ij1} + \\
& + \sum_{i=1}^{n_2} \sum_{j=1}^{m_2} (f_{ij2} - 1)^2 \, a_{ij2}^{\,2} / g_{ij2} + \\
& - \sum_{i=1}^{n_1} \lambda_{i1} \sum_{j=1}^{m_1} f_{ij1} \, a_{ij1} - \sum_{i=1}^{n_2} \lambda_{i2} \sum_{j=1}^{m_2} f_{ij2} \, a_{ij2} + \\
& - \sum_{j=1}^{m_1} \mu_{j1} \sum_{i=1}^{n_1} f_{ij1} \, a_{ij1} - \sum_{j=1}^{m_2} \mu_{j2} \sum_{i=1}^{n_2} f_{ij2} \, a_{ij2} + \\
& - \sum_{h=1}^{v_r} \nu_{h1} (f_{hm_1 1} \, a_{hm_1 1} - f_{hm_2 2} \, a_{hm_2 2}) + \\
& - \nu_{12} (f_{n_1 m_1 1} \, a_{n_1 m_1 1} - f_{n_2 m_2 2} \, a_{n_2 m_2 2}) + \\
& - \sum_{h=1}^{v_c} \nu_{h3} (f_{n_1 h1} \, a_{n_1 h1} - f_{n_2 h2} \, a_{n_2 h2})
\end{aligned}
\qquad (71)
$$

Here, multiplier ν_{12} makes sure that the total-general of both subtables is equal.

This leads to the following partial derivatives:

$$\frac{\partial L}{\partial f_{klx}} = (f_{klx} - 1) \; a_{klx}{}^{2} \; / \; g_{klx} - \lambda_{kx} \; a_{klx} - \mu_{lx} \; a_{klx} +$$
$$- \nu_{kl} \; (a_{klx} \; \delta_{lm_1} \; \delta_{x1} - a_{klx} \; \delta_{lm_2} \; \delta_{x2}) \; \eta_{kv_r} +$$
$$- \nu_{12} \; (a_{klx} \; \delta_{kn_1} \; \delta_{lm_1} \; \delta_{x1} - a_{klx} \; \delta_{kn_2} \; \delta_{lm_2} \; \delta_{x2}) +$$
$$- \nu_{13} \; (a_{klx} \; \delta_{kn_1} \; \delta_{x1} - a_{klx} \; \delta_{kn_2} \; \delta_{x2}) \; \eta_{lv_c} \qquad (72)$$

which give the following equation

$$\{ \lambda_{kx} + \mu_{lx} + \nu_{kl} \; (\delta_{lm_1} \; \delta_{x1} - \delta_{lm_2} \; \delta_{x2}) \; \eta_{kv_r} +$$
$$+ \nu_{12} \; (\delta_{kn_1} \; \delta_{lm_1} \; \delta_{x1} - \delta_{kn_2} \; \delta_{lm_2} \; \delta_{x2})$$
$$+ \nu_{13} \; (\delta_{kn_1} \; \delta_{x1} - \delta_{kn_2} \; \delta_{x2}) \; \eta_{lv_c} \; \} \; g_{klx} = (f_{klx} - 1) \; a_{klx} \qquad (73)$$

Substitution of (73) in (49) and (5) give the following equations to be solved

$$\sum_{j=1}^{m_x} \{ \lambda_{kx} + \mu_{jx} + \nu_{kl} \; (\delta_{jm_1} \; \delta_{x1} - \delta_{jm_2} \; \delta_{x2}) \; \eta_{kv_r} +$$
$$+ \nu_{12} \; (\delta_{kn_1} \; \delta_{jm_1} \; \delta_{x1} - \delta_{kn_2} \; \delta_{jm_2} \; \delta_{x2}) +$$
$$+ \nu_{j3} \; (\delta_{kn_1} \; \delta_{x1} - \delta_{in_2} \; \delta_{x2}) \; \eta_{jv_c} \; \} \; g_{kjx} = s_{kx} \qquad (74)$$

$$\sum_{i=1}^{n_x} \{ \lambda_{ix} + \mu_{lx} + \nu_{il} \; (\delta_{lm_1} \; \delta_{x1} - \delta_{lm_2} \; \delta_{x2}) \; \eta_{iv_r} +$$
$$+ \nu_{12} \; (\delta_{in_1} \; \delta_{lm_1} \; \delta_{x1} - \delta_{in_2} \; \delta_{lm_2} \; \delta_{x2}) +$$
$$+ \nu_{13} \; (\delta_{in_1} \; \delta_{x1} - \delta_{in_2} \; \delta_{x2}) \; \eta_{lv_c} \; \} \; g_{ilx} = d_{lx} \qquad (75)$$

Equation (73) should subsequently be substituted in (70). As v_r is smaller than n_1 or n_2 and as v_c is smaller than m_1 or m_2, the products of the Kronecker deltas in (72) are zero. The substitution therefore leads to

$$\lambda_{k1}\, g_{km_1 1} - \lambda_{k2}\, g_{km_2 2} + \mu_{m_1 1}\, g_{km_1 1} - \mu_{m_2 2}\, g_{km_2 2} + \\ + \nu_{k1}\, (\, g_{km_1 1} + g_{km_2 2}\,) = a_{km_2 2} - a_{km_1 1} \qquad (k \le v_r) \qquad (76a)$$

$$\lambda_{n_1 1}\, g_{n_1 m_1 1} - \lambda_{n_2 2}\, g_{n_2 m_2 2} + \mu_{m_1 1}\, g_{n_1 m_1 1} - \mu_{m_2 2}\, g_{n_2 m_2 2} + \\ + \nu_{12}\, (\, g_{n_1 m_1 1} + g_{n_2 m_2 2}\,) = a_{n_1 m_1 1} - a_{n_2 m_2 2} \qquad (76b)$$

$$\lambda_{n_1 1}\, g_{n_1 11} - \lambda_{n_2 2}\, g_{n_2 12} + \mu_{11}\, g_{n_1 11} - \mu_{12}\, g_{n_2 12} + \\ + \nu_{13}\, (\, g_{n_1 11} + g_{n_2 12}\,) = a_{n_2 12} - a_{n_1 11} \qquad (k \le v_c) \qquad (76c)$$

Of the above equations, (74) leads to $n = n_1 + n_2$ equations, (75) to $m = m_1 + m_2$ equations and (76) to $v = v_r + v_c$ equations. In most cases, $n < v < m$ so that a solution to this simultaneous system must be sought in an analogy with the matrix equation (53) so that the special structure can be used for calculating Ω. We then get

$$\left[\begin{array}{cc|ccc|cc} \hat{p}_1 & 0 & X_{11} & X_{12} & X_{13} & G_1 & 0 \\ 0 & \hat{p}_2 & X & X_{22} & X_{23} & 0 & G_2 \\ \hline X_{11}' & X_{21}' & \hat{z}_1 & 0 & 0 & Y_{11} & Y_{12} \\ X_{12}' & X_{22}' & 0 & \hat{z} & 0 & Y_{21} & Y_{22} \\ X_{13}' & X_{23}' & 0 & 0 & \hat{z} & Y_{31} & Y_{32} \\ \hline G_1' & 0 & Y_{11}' & Y_{21}' & Y_{31}' & \hat{q}_1 & 0 \\ 0 & G_2' & Y_{12}' & Y_{22}' & Y_{32}' & 0 & \hat{q}_2 \end{array}\right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \hline \nu_1 \\ \nu_2 \\ \nu_3 \\ \hline \mu_1 \\ \mu_2 \end{array}\right] = \left[\begin{array}{c} s_1 \\ s_2 \\ \hline u_1 \\ u_2 \\ u_3 \\ \hline d_1 \\ d_2 \end{array}\right] \qquad (77)$$

The elements of (77) are defined as follows. G_1 and G_2 are the weight matrices belonging to A_1 and A_2 respectively; the dimensions are thus $n_1 \times m_1$ and $n_2 \times m_2$ respectively. Diagonal matrices $\hat{p}_1$ and $\hat{p}_2$ are the respective row sums of the weight matrices G_1 and G_2, while both $\hat{q}_1$ and $\hat{q}_2$ contain the corresponding column sums.

From (76a) it follows that elements z_{1kk} of $\hat{z}_1$ are given by

$$z_{1kk} = g_{k\ m_1 1} + g_{k\ m_2 2} \tag{78a}$$

Matrix $\hat{z}_2$ consists of only one element, z_{211}, whose value follows from (76b)

$$z_{211} = g_{n_1 m_1 1} + g_{n_2 m_2 2} \tag{78b}$$

The elements of $\hat{z}_3$ follow from (76c)

$$z_{311} = g_{n_1 1\ 1} + g_{n_2 1\ 2} \tag{78c}$$

Matrices X_{ij} are sparse matrices with a special structure. Matrices X_{i1} contain only nonzero elements diagonally from the top left-hand corner:

$$x_{11k\ k} = g_{k\ m_1 1} \qquad (\ k \le v_r\ ) \tag{79a}$$

$$x_{21k\ k} = -\ g_{k\ m_2 2} \qquad (\ k \le v_r\ ) \tag{79b}$$

Matrices X_{i2} are in fact vectors; only the bottom element is nonzero

$$x_{13n_1 1} = g_{n_1 m_1 1} \tag{79c}$$

$$x_{23n_2 1} = -\ g_{n_2 m_2 2} \tag{79d}$$

Matrices X_{i3} contain only nonzero elements on the bottom row

$$x_{12n_1 1} = g_{n_1 1\ 1} \qquad (\ 1 \le v_c\ ) \tag{79e}$$

$$x_{22n_2 1} = -\ g_{n_2 1\ 2} \qquad (\ 1 \le v_c\ ) \tag{79f}$$

Matrices Y_{ij} are also sparse matrices, the forms of which are comparable with those in X_{ij}. From (76) we find that

$$y_{11km_1} = g_{k\ m_1 1} \qquad (\ k \le v_r\ ) \qquad (80a)$$

$$y_{12km_2} = -\ g_{k\ m_2 2} \qquad (\ k \le v_r\ ) \qquad (80b)$$

$$y_{211m_1} = g_{n_1 m_1 1} \qquad (80c)$$

$$y_{221m_2} = -\ g_{n_2 m_2 2} \qquad (80d)$$

$$y_{3111} = g_{n_1 1\ 1} \qquad (\ 1 \le v_c\ ) \qquad (80e)$$

$$y_{3211} = -\ g_{n_2 1\ 2} \qquad (\ 1 \le v_c\ ) \qquad (80f)$$

The elements of subvectors u_h on the right-hand side of (77) follow from (76):

$$u_{1k} = a_{k\ m_2 2} - a_{k\ m_1 1} \qquad (\ k \le v_r\ ) \qquad (81a)$$

$$u_{21} = a_{n_1 m_1 1} - a_{n_2 m_2 2} \qquad (81b)$$

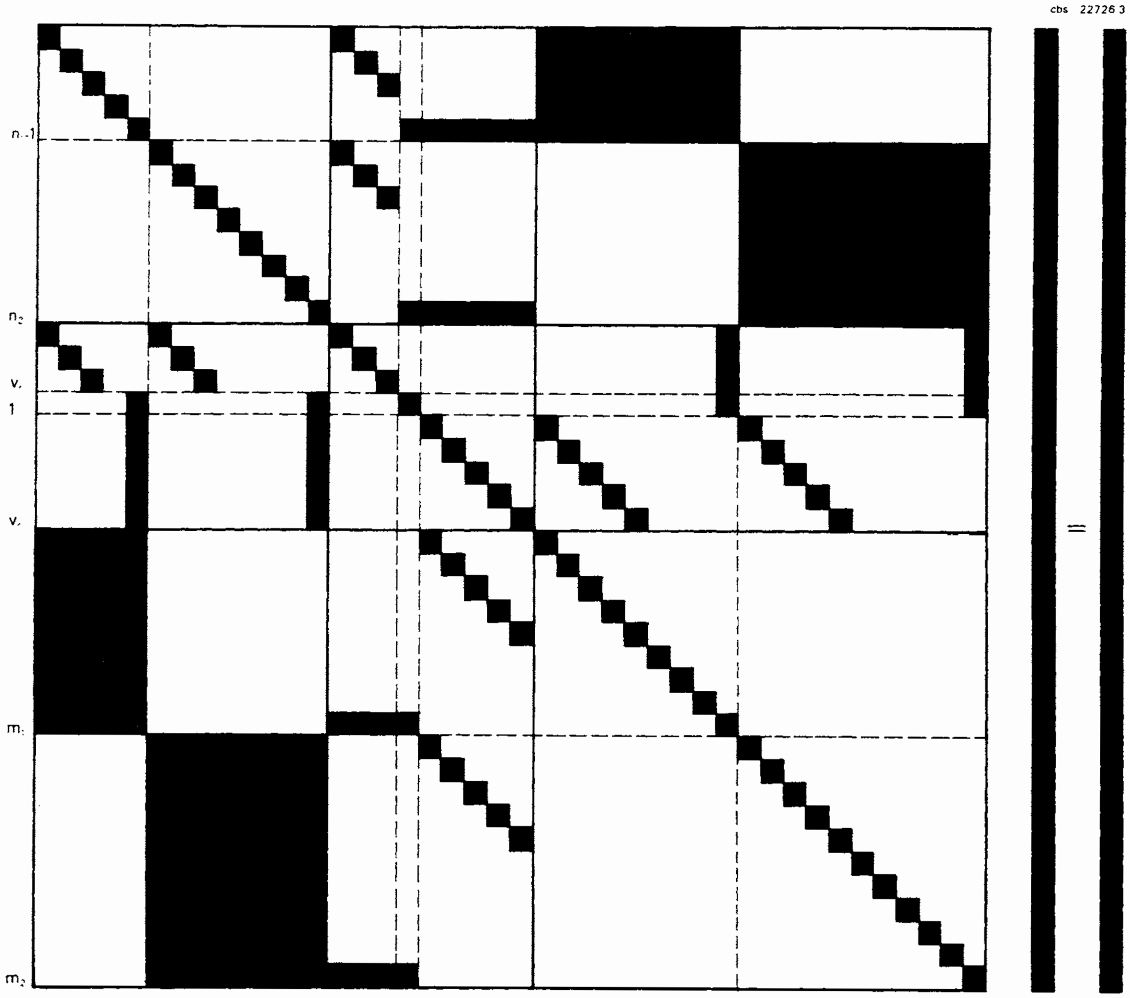
$$u_{31} = a_{n_2 1\ 2} - a_{n_1 1\ 1} \qquad (\ 1 \le v_c\ ) \qquad (81c)$$

Figure 3 is a schematic representaion of equation (77). The composition of equation (77) is comparable with that of equation (53); this is also obvious from the rough similarity between figures 1 and 2. We can solve system (77) by the same methods used for system (53). This means that we should carry out the same steps as (53) - (60). This reduces the system of equations to a number of multiplications, an inverse of the order ($v_r + v_c + 1$) and the solution of a considerably reduced system of the order ($n - 1$). By employing the knowledge about which parts of the submatrices are zero by definition, the number of steps can be greatly reduced. In addition, the calculation of the term Ω can be simplified.

Just as in chapter 8, here Ω has the same form as in (57), but here J is a identity matrix.

3. **The pattern of equation (77); the dimensions are mentioned on the left-hand side**

$n_1-1=5$, $m_1=9$, $n_2=8$, $m_2=11$, $v_1=3$, $V_2=5$



$$\Omega = (I - F F')^{-1}$$

$$= \left\{ \begin{bmatrix} I_{v_r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{v_c} \end{bmatrix} + \right.$$

$$\left. - \begin{bmatrix} F_{11}F_{11}'+F_{12}F_{12}' & F_{11}F_{21}'+F_{12}F_{22}' & F_{11}F_{31}'+F_{12}F_{32}' \\ F_{21}F_{11}'+F_{22}F_{12}' & F_{21}F_{21}'+F_{22}F_{22}' & F_{21}F_{31}'+F_{22}F_{32}' \\ F_{31}F_{11}'+F_{32}F_{12}' & F_{31}F_{21}'+F_{32}F_{22}' & F_{31}F_{31}'+F_{32}F_{32}' \end{bmatrix} \right\}^{-1}$$

$$= \left\{ \begin{bmatrix} I_{v_r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{v_c} \end{bmatrix} - \left[\begin{array}{cc|c} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ \hdashline B_{31} & B_{32} & B_{33} \end{array} \right] \right\}^{-1} \qquad (82)$$

It is relatively easy to see from figure 3 that a number of products of submatrices in (82) result in zero-matrices, so that (82) can also be written as

$$\Omega = \left\{ \begin{bmatrix} I_{v_r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{v_c} \end{bmatrix} - \left[\begin{array}{cc|c} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ \hdashline 0 & 0 & B_{33} \end{array} \right] \right\}^{-1}$$

$$= \left[\begin{array}{cc|c} I_{v_r} - B_{11} & B_{12} & 0 \\ B_{21} & 1 - B_{22} & 0 \\ \hdashline 0 & 0 & I_{v_c} - B_{33} \end{array} \right]^{-1} \qquad (83)$$

The elements of the submatrices can be determined by combining equations (82), (53), (80) and (78), which results in:

$$b_{11ij} = (g_{im_1 1}\, g_{jm_1 1}\, q_{1m_1 m_1}^{-1} + g_{im_2 2}\, g_{jm_2 2}\, q_{2m_2 m_2}^{-1}) \cdot$$
$$\cdot (g_{im_1 1} + g_{im_2 2})^{-\frac{1}{2}} (g_{jm_1 1} + g_{jm_2 2})^{-\frac{1}{2}} \tag{84a}$$

$$b_{12i1} = (g_{im_1 1}\, g_{n_1 m_1 1}\, q_{1m_1 m_1}^{-1} + g_{im_2 2}\, g_{n_2 m_2 2}\, q_{2m_2 m_2}) \cdot$$
$$\cdot (g_{im_1 1} + g_{im_2 2})^{-\frac{1}{2}} (g_{n_1 m_1 1} + g_{n_2 m_2 2})^{-\frac{1}{2}} \tag{84b}$$

$$b_{211j} = b_{12j1} \tag{84c}$$

$$b_{2211} = (g_{n_1 m_1 1}^{2}\, q_{1m_1 m_1}^{-1} + g_{n_2 m_2 2}^{2}\, q_{2m_2 m_2}^{-1}) (g_{n_1 m_1 1} + g_{n_2 m_2 2})^{-1} \tag{84d}$$

$$b_{33ii} = (g_{n_1 i1}^{2}\, q_{1ii}^{-1} + g_{n_2 i1}^{2} + q_{2ii}^{-1}) (g_{n_1 i1} + g_{n_2 i2})^{-1} \tag{84e}$$

As the bottom right-hand submatrix of (83) is a diagonal matrix, its inverse can be easily calculated. In order to calculate Ω we now only have to invert the symmetrical top left-hand matrix of (83). The calculation of Ω therefore requires only the inversion of a symmetrical matrix of the order ($v_r + 1$).

This chapter has demonstrated how the original system of equations, which is of the order ($n_1 + n_2 + v_r + 1 + v_c + m_1 + m_2$), can be reduced to a symmetrical system of the order ($n_1 + n_2$). To show just how important this is, let us take a not unrealistically large example. Say $n_1 = n_2 = 300$, $m_1 = m_2 = 2500$, $v_r = 250$ and $v_c = 2350$. The original system then contains 7700 equations with just as many unknowns; such a system is hardly manageable, even by today's super computers. Having applied the methods described here, all that remains to be solved is a symmetrical system of the order $n_1 + n_2 = 600$; such a system may be considered large, but it can be solved within a reasonable time by a modern mainframe computer.

10. Three-dimensional tables

Three-dimensional tables differ from two-dimensional ones in that the edges are planes instead of lines. When the row and column sums of a two-dimensional table are known, three planes with the row and column sums concerned will constitute a set of constraints. These three planes can, with a negative sign, be stuck on to the three dimensional table. The missing ribs can be filled with sums of the planes at right angles to the rib (with unchanged sign); the missing corner is filled with the total-general of the whole table (again with the opposite sign). In this extended table, the sum of the elements in a bar at right angles to an edge should equal nul. In practice it often turns out that this requirement is not met. With the aid of Lagrange multipliers, factors $f_{i_1 i_2 i_3}$ can be determined in such a way that after multiplying the corresponding elements of the table by these factors (a three-dimensional Hadamar product), a new, 'minimally changed' table is created which does meet the requirements concerned.

For an (m_1 x m_2 x m_3) table A with elements $a_{i_1 i_2 i_3}$, factors $f_{i_1 i_2 i_3}$ should be calculated in such a way that

$$\left.\begin{aligned} \sum_{i_1=1}^{m_1} f_{i_1 k_2 k_3}\, a_{i_1 k_2 k_3} &= 0 \\ \sum_{i_2=1}^{m_2} f_{k_1 i_2 k_3}\, a_{k_1 i_2 k_3} &= 0 \\ \sum_{i_3=1}^{m_3} f_{k_1 k_2 i_3}\, a_{k_1 k_2 i_3} &= 0 \end{aligned}\right\} \quad \begin{aligned} k_1 &\le m_1 \\ k_2 &\le m_2 \\ k_3 &\le m_3 \end{aligned} \qquad (85)$$

The Lagrangian now takes the following form

$$L = \frac{1}{2} \sum_{i_1=1}^{m_1} \sum_{i_2=2}^{m_2} \sum_{i_3=1}^{m_3} (f_{i_1 i_2 i_3} - 1)^2 a_{i_1 i_2 i_3}{}^2 / g_{i_1 i_2 i_3} +$$

$$- \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \lambda^3_{i_1 i_2} \sum_{i_3=1}^{m_3} f_{i_1 i_2 i_3} a_{i_1 i_2 i_3} +$$

$$- \sum_{i_1=1}^{m_1} \sum_{i_3=1}^{m_3} \lambda^2_{i_1 i_3} \sum_{i_2=1}^{m_2} f_{i_1 i_2 i_3} a_{i_1 i_2 i_3} +$$

$$- \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} \lambda^1_{i_2 i_3} \sum_{i_1=1}^{m_1} f_{i_1 i_2 i_3} a_{i_1 i_2 i_3} = \min. \tag{86}$$

The only difference with (7) is that there are now 3 sets of Lagrange multipliers. The further caculations are completely analogous.

$$\frac{\partial L}{\partial f_{k_1 k_2 k_3}} = (f_{k_1 k_2 k_3} - 1) a_{k_1 k_2 k_3}{}^2 / g_{k_1 k_2 k_3} +$$

$$- (\lambda^3_{k_1 k_2} + \lambda^2_{k_1 k_3} + \lambda^1_{k_2 k_3}) a_{k_1 k_2 k_3} = 0 \tag{87}$$

and therefore

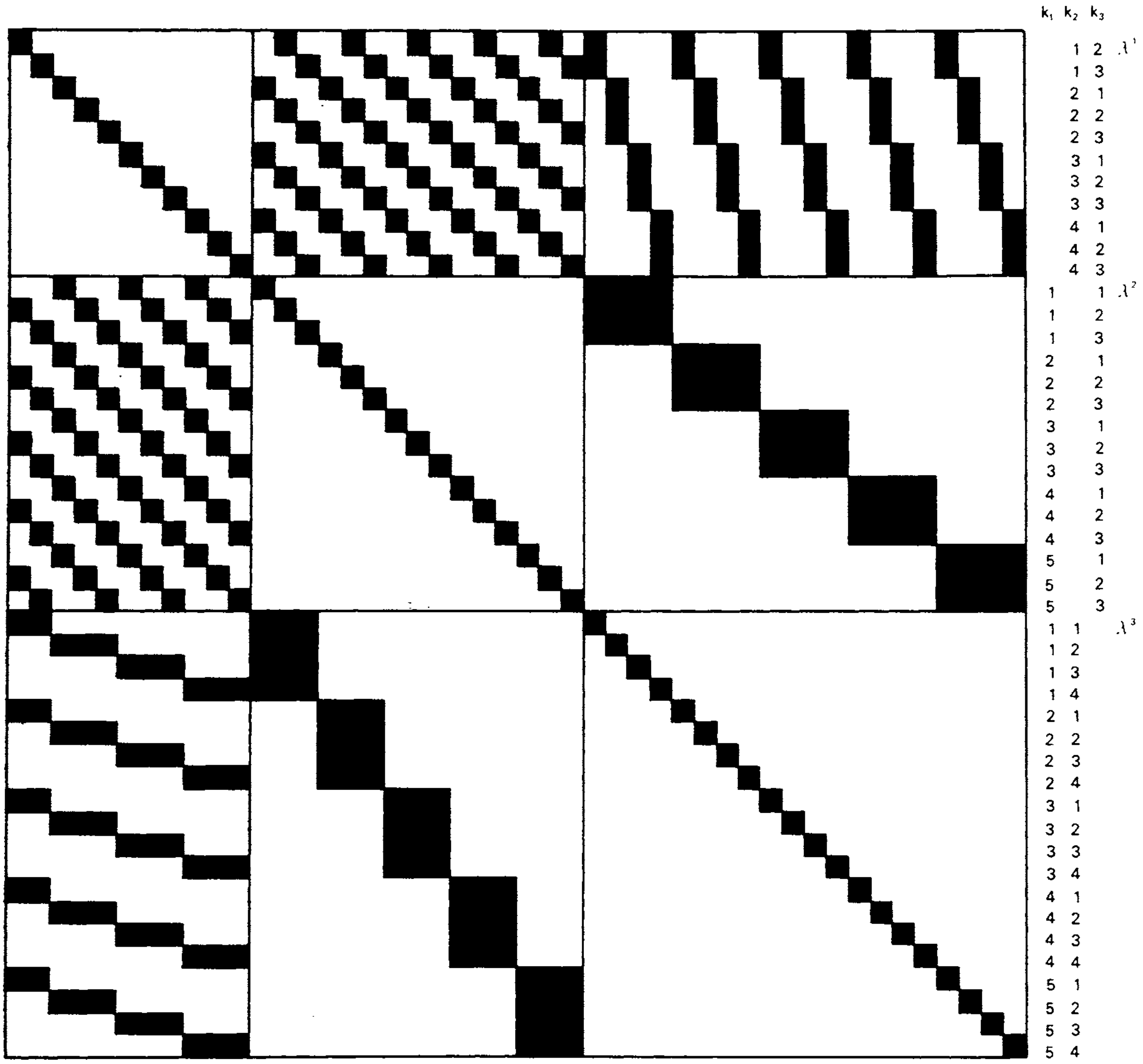
$$(\lambda^1_{k_2 k_3} + \lambda^2_{k_1 k_3} + \lambda^3_{k_1 k_2}) g_{k_1 k_2 k_3} = (f_{k_1 k_2 k_3} - 1) a_{k_1 k_2 k_3} \tag{88}$$

Substituting (88) in (85) we get the simultaneous system of equations (89)

$$\begin{aligned}
\sum_{i_1=1}^{m_1} (\lambda^1_{k_2 k_3} + \lambda^2_{i_1 k_3} + \lambda^3_{i_1 k_2}) g_{i_1 k_2 k_3} &= - \sum_{i_1=1}^{m_1} a_{i_1 k_2 k_3} \\
\sum_{i_2=1}^{m_2} (\lambda^1_{i_2 k_3} + \lambda^2_{k_1 k_3} + \lambda^3_{k_1 i_2}) g_{k_1 i_2 k_3} &= - \sum_{i_2=1}^{m_2} a_{k_1 i_2 k_3} \\
\sum_{i_3=1}^{m_3} (\lambda^1_{k_2 i_3} + \lambda^2_{k_1 i_3} + \lambda^3_{k_1 k_2}) g_{k_1 k_2 i_3} &= - \sum_{i_3=1}^{m_3} a_{k_1 k_2 i_3}
\end{aligned} \tag{89}$$

These equations can be transformed into a matrix equation

4. **Matrix W and the construction of vector λ.**
The first row and column of W have been removed to adjust the order of W to its rank
$m_1=5$, $M_2=4$, $m_3=3$



$$W \lambda = s \tag{90}$$

where vector λ is composed of subvectors λ^1, λ^2 and λ^3. Figure 4 shows where the nonzero elements of matrix W can be found.
We can consider matrix W as a 3 x 3 -matrix with submatrices $W_{j_1 j_2}$ as elements. Each of these submatrices can in turn be considered as being

built up of $l_1 \times l_2$ sub-submatrices. For elements $w_{j_1 j_2 l_1 l_2 h_1 h_2}$ in W

$$
\begin{aligned}
&w_{11 l_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \quad \vee \quad h_1 \neq h_2 \\
&w_{11 l\, l\, h\, h} = \sum_{i_1=1}^{m_1} g_{i_1 l h} && \\
&w_{12 l_1 l_2 h_1 h_2} = 0 && h_1 \neq h_2 \\
&w_{12 l_1 l_2 h\, h} = g_{l_2 l_1 h} && \\
&w_{13 l_1 l_2 h_1 h_2} = 0 && l_1 \neq h_2 \\
&w_{13 l_1 l_2 h_1 l_1} = g_{l_2 l_1 h_1} && \qquad\qquad (91) \\
&w_{22 l_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \quad \vee \quad h_1 \neq h_2 \\
&w_{22 l\, l\, h\, h} = \sum_{i_2=1}^{m_2} g_{l i_2 h} && \\
&w_{23 l_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \\
&w_{23 l\, l\, h_1 h_2} = g_{l h_2 h_1} && \\
&w_{33 l_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \quad \vee \quad h_1 \neq h_2 \\
&w_{33 l\, l\, h\, h} = \sum_{i_3=1}^{m_3} g_{l h i_3} &&
\end{aligned}
$$

As matrix W is symmetrical, the whole matrix is determined by (91). To solve equation (90) the method in chapter 7 from equation (53) onwards can again be used. For the elements of the transformation diagonal matrix we choose

$$\tau_{jjllhh} = w_{jjllhh}^{\frac{1}{2}} \qquad (92)$$

and construct

$$M = \hat{\tau}^{-1} \, W \, \hat{\tau}^{-1} \tag{93}$$

and

$$\xi = \hat{\tau} \, \lambda \tag{94}$$

and

$$\alpha = \hat{\tau}^{-1} \, s \tag{95}$$

respectively. The system now to be solved is

$$M \, \xi = \alpha \tag{96}$$

of which the form is comparable with (36). The diagonal of M consists only of ones.

As stated in chapter 7, calculations can be reduced by choosing an order in which the largest submatrix after partitioning is inverted first, followed by the next largest. To link up with the equations in chapter 7, the largest matrix must therefore be placed at the bottom right; this has already been done in figure 4. Indices k_1, k_2 and k_3 of $g_{k_1 k_2 k_3}$ must be chosen in such an order that $m_1 \geq m_2 \geq m_3$.

The partitioned solution to (96) was given in (31) and (53) - (60), be it that here matrix J is a unit matrix I. The original system, which was of the order ($m_1 \times m_2 + m_1 \times m_3 + m_2 \times m_3$), has now been replaced by an inverse of the matrix Ω which is of the order ($m_2 \times m_3$). Although the orders of the problems have been reduced enormously, the new orders may still involve an insurmountable amount of calculation. Another view of Ω shows, however, that the problem can be further reduced quite simply:

$$\Omega^{-1} = I - F F'$$

$$= I - M_{23} M_{32} \tag{97}$$

Due to the special form of F (see figure 4), Ω^{-1} turns out to take the form of a block-diagonal matrix in the three-dimensional case. The matrix Ω^{-1} consists of m_2 square matrices of the order m_3. Elements

$$\omega_{l_1 l_2 h_1 h_2} = 0 \qquad l_1 \neq l_2 \tag{98a}$$

$$\omega_{l\,l\,h_1 h_2} = \delta_{h_1 h_2} - \sum_{h=1}^{m_1} f_{llh_1 h}\, f_{llh_2 h}$$

$$= \delta_{h_1 h_2} - \sum_{h=1}^{m_1} g_{lhh_1}\, g_{lhh_2} \tag{98b}$$

5. **The pattern of both matrix Ω and its inverse Ω^{-1}**
m_1=5, m_3=3

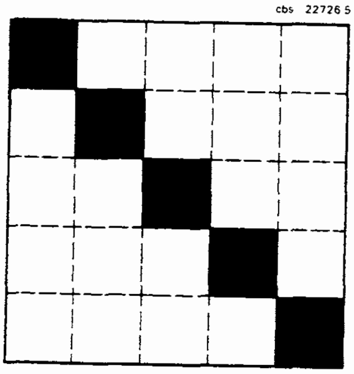


Figure 5 shows the form of matrix Ω^{-1}. Due to its special form, the form of the inverse Ω is identical. To calculate Ω we can 'suffice' by determining the m_1 inverses of the m_3 x m_3 -submatrices Ω_{11}^{-1}.

Eventually, only a system similar to the one in (55) remains to be solved. In this case it is a system of the order (m_2 x m_3).

By way of example of a three-dimensional table, let us take a labour table with 79 occupational categories, 29 demographic characteristics and a regional classification into 12 areas. This results in a 80 x 30 x 13 -table. Apart from various matrix multiplications, the main calculation work consists of the determination of 80 inverses of symmetrical 13 x 13 -matrices and the solution to a system with a symmetrical 390 x 390 -matrix. If the regional classification is increased from 12 to 44 areas, the table should be overturned to an 80 x 45 x 30 -table. Now 80 symmetrical matrices of 30 x 30 have to be inverted and a system containing a symmetrical matrix of 1350 x 1350 be solved.

Neglecting the computer IO operations, the method using the Lagrange multipliers will generally take many times the caculation time necessary for the RAS method. The calculation time with the Lagrange multipliers is dominated by the largest matrix to be inverted, which has dimensions m_1 x m_2; the number of operations (multiplications, additions, equations) is then approximately $\frac{1}{2}$ (m_2 x m_3 $)^3$ + $\frac{1}{2}$ m_1 $m_3{}^3$. For the RAS method, the number of operations is approxiamtely 3 i m_1 m_2 m_3, where i is the number of iteration steps. Only when $m_1 >> m_2{}^3$ do the Lagrange multipliers take less computation time than the RAS method, if 6 i $m_2 > m_3{}^2$. This situation occurs with very long flat tables ('shelves'). For that matter, the method with the Lagrange multipliers may offer advantages sooner if modern-day computers are used for the processing. In situations where not all the elements of the table fit into the central memory simultaneously, part of the data have to be read in three times for each iteration in the RAS method. If the symmetrical system of the order (m_2 x m_3) does fit into the central memory, computer use can be cut back drastically due to the much lower number of IO operations. So, if we do not neglect the IO operations, the Lagrange multipliers method may also be more advatageous than the RAS method with respect to computer use.

Another advantage of Lagrange multipliers is that uncertainties in the calculations of the various elements can be taken into account, whereby uncertainties are also permitted in the row and column sums.

11. Identical row and column sums in three-dimensional tables

We have already seen in chapter 8 that in compiling input-output tables there is often the requirement that a number of column sums should equal the row sums when the row an column numbers are the same. In the case of regional input-output tables, this requirement is valid for every region, while the sum of all the regions of a certain cell must result in the national totals concerned. What we then get, therefore, is a three-dimensional table with in one direction (perpendicular to the planes of the regional tables) column sums identical to the row totals for a certain plane.

The situation described above leads to an extra restraint to those formulated in (85)

$$f_{hm_2k_3}\ a_{hm_2k_3} - f_{m_1hk_3}\ a_{m_1hk_3} = 0 \qquad h \le v < m_2 \le m_1 \qquad (99)$$

This means that a term has to be added to Lagrangian L in (86); the new Lagrangian L now becomes

$$\Gamma = L - \sum_{h=1}^{v} \sum_{i_3=1}^{m_3} \nu_{hi_3}\ (\ f_{hm_2i_3}\ a_{hm_2i_3} - f_{m_1hi_3}\ a_{m_1hi_3}\ ) = \min. \qquad (100)$$

The partial derivative now becomes

$$\frac{\partial \Gamma}{\partial f_{k_1k_2k_3}} = \frac{\partial L}{\partial f_{k_1k_2k_3}} - \nu_{k_1k_3}\ a_{k_1k_2k_3}\ \eta_{k_1v}\ \delta_{k_2m_2} +$$

$$+ \nu_{k_2k_3}\ a_{k_1k_2k_3}\ \eta_{k_2v}\ \delta_{k_1m_1} = 0 \qquad (101)$$

Equation (101) can be substituted in (85) and (90), resulting in a system of equations

$$\sum_{i_1=1}^{m_1}(\lambda^1_{k_2k_3} + \lambda^2_{i_1k_3} + \lambda^3_{i_1k_2} + \nu_{i_1k_3}\ \eta_{k_1v}\ \delta_{k_2m_2} - \nu_{k_2k_3}\ \eta_{k_2v}\ \delta_{i_1m_1})\ g_{i_1k_2k_3} =$$

$$= - \sum_{i_1=1}^{m_1} a_{i_1k_2k_3}$$

$$\sum_{i_2=1}^{m_2}(\lambda^1_{i_2k_3} + \lambda^2_{k_1k_3} + \lambda^3_{k_1i_2} + \nu_{k_1k_3}\ \eta_{k_1v}\ \delta_{i_2m_2} - \nu_{i_2k_3}\ \eta_{i_2v}\ \delta_{k_1m_1})\ g_{k_1i_2k_3} =$$

$$= - \sum_{i_2=1}^{m_2} a_{k_1i_2k_3}$$

$$\sum_{i_3=1}^{m_3}(\lambda^1_{k_2i_3} + \lambda^2_{k_1i_3} + \lambda^3_{k_1k_2} + \nu_{k_1i_3}\ \eta_{k_1v}\ \delta_{k_2m_2} - \nu_{k_2i_3}\ \eta_{k_2v}\ \delta_{k_1m_1})\ g_{k_1k_2i_3} = \tag{102}$$

$$= - \sum_{i_3=1}^{m_3} a_{k_1k_2i_3}$$

$$(\lambda^1_{m_2k_3} + \lambda^2_{hk_3} + \lambda^3_{hm_2} + \nu_{hk_3})\ g_{hm_2k_3} - (\lambda^1_{hk_3} + \lambda^2_{m_1k_3} + \lambda^3_{m_1h} - \nu_{hk_3})\ g_{m_1hk_3} =$$

$$= a_{m_1hk_3} - a_{hm_2k_3}$$

This system can also be written in the form of (90). It now proves convenient to construct vector λ from λ^1, ν, λ^2 and λ^3 successively. Here, therefore, matrix W is a 4 x 4 -matrix of submatrices $W_{j_1j_2}$. If we combine the second and third rows (and therefore also the second and third columns), the solution by means of partitioning is similar to that of 3 x 3 submatrices. The indices of j_1 and j_2 are numbered 1, 2^1, 2^2, 3 accordingly. The composition of matrix W is identical to that in equation (91) in chapter 10 for corresponding indices j_1 and j_2 where index 2^2 here corresponds with 2 in (91). The elements of submatrices $W_{2^1j_2}$ and $W_{j_12^1}$ now remain to be determined. In view of the symmetry, only the elements of the top triangle of W are given by (103)

$$
\begin{aligned}
&w_{1\,2^1 l_1 l_2 h_1 h_2} = 0 && (\, l_1 \neq l_2 \;\wedge\; l_1 < m_2 \,) \;\vee\; h_1 \neq h_2 \\
&w_{1\,2^1 l\,l\,h\,h} = -\, g_{m_1 lh} && \\
&w_{1\,2^1 m_2 l\,h\,h} = g_{lm_2 h} && \\
&w_{2^1 2^1 l_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \;\vee\; h_1 \neq h_2 \\
&w_{2^1 2^1 l\,l\,h\,h} = g_{lm_2 h} + g_{m_1 lh} && \\
&w_{2^1 2^2 l_1 l_2 h_1 h_2} = 0 && (\, l_1 \neq l_2 \;\wedge\; l_2 < m_1 \,) \;\vee\; h_1 \neq h_2 \\
&w_{2^1 2^2 l\,l\,h\,h} = g_{lm_2 h} && \qquad (103) \\
&w_{2^1 2^2 l\,m_1 h\,h} = -\, g_{m_1 lh} && \\
&w_{2^1 3\, l_1 l_2 h_1 h_2} = 0 && (\, l_1 \neq l_2 \;\vee\; h_2 < m_2 \,) \wedge (\, l_1 < m_2 \;\vee\; h_2 \neq l_1 \,) \\
&w_{2^1 3\, l\,l\,h\,m_2} = g_{lm_2 h} && \\
&w_{2^1 3\, l\,m_1 h\,l} = -\, g_{m_1 lh} &&
\end{aligned}
$$

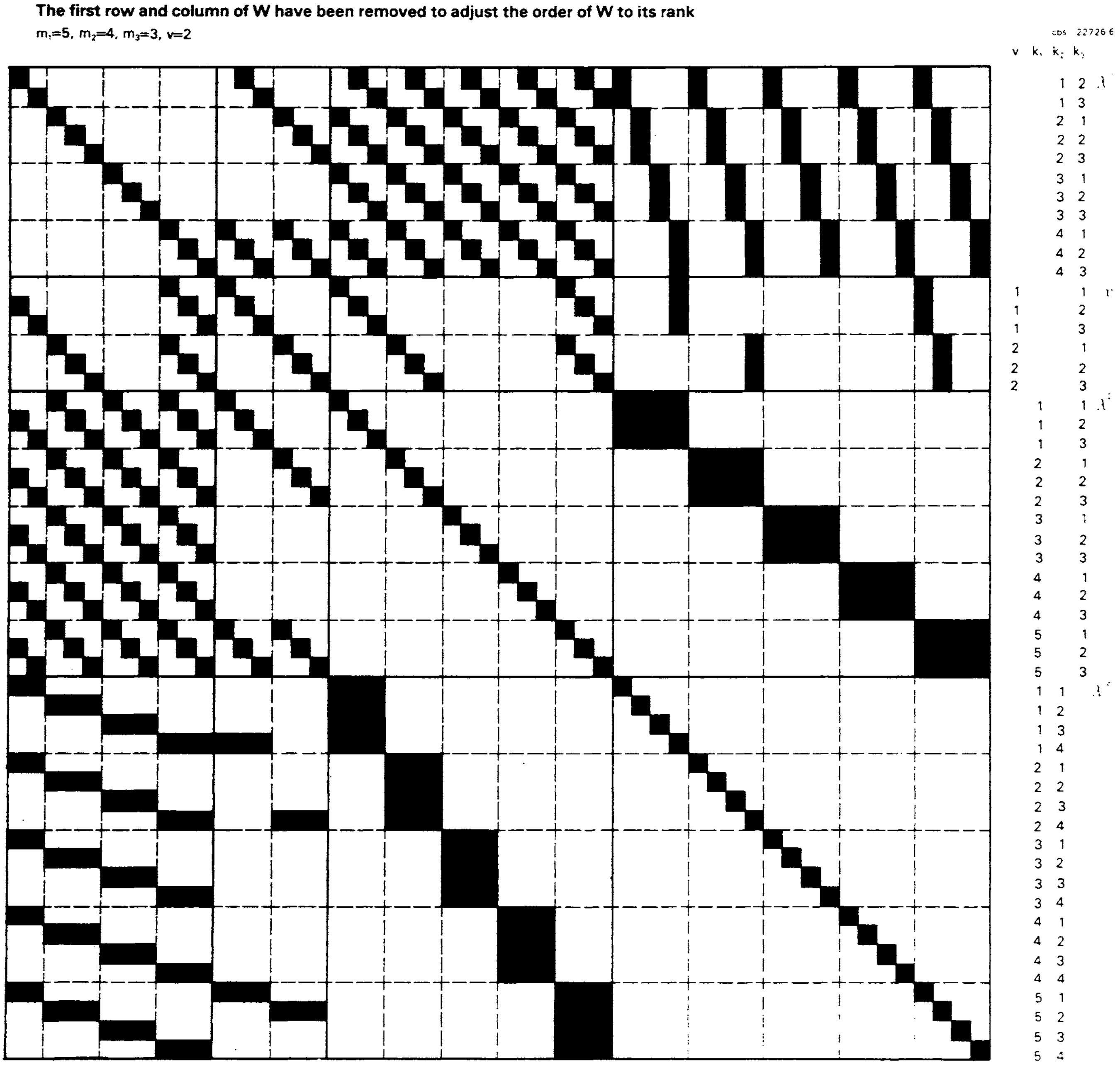
The position of the nonzero elements in W are shown schematically in figure 6.

The further calculation is completely analogous with (92) - (96). By partitioning matrix M into 4 x 3 submatrices, the solution as presented in (31) and (54) - (60) can once again be used. As opposed to chapter 10, J is not a unit matrix here. Here, system (96) is of the order ($m_2 \times m_3 + v \times m_3 + m_1 \times m_3 + m_2 \times m_3$). Via the partitioning process we are now confronted with the calculation of the inverse of a matrix of the order ($v \times m_3 + m_1 \times m_3$), from which the solution of a system of the order $m_2 \times m_3$ will result. In many cases, modern-day computers will hardly be able to cope with the calculation of Ω, which requires an inverse of the order ($v \times m_3 + m_1 \times m_3$). However, once again partitioning can greatly simplify this process.

6. Matrix W and the construction of vector λ

The first row and column of W have been removed to adjust the order of W to its rank

m_1=5, m_2=4, m_3=3, v=2



If we write Ω^{-1} in (31) in the partitioned form, we find

$$\left[\begin{array}{c|c} (\Omega^{-1})_{11} & (\Omega^{-1})_{12} \\ \hline (\Omega^{-1})_{21} & (\Omega^{-1})_{22} \end{array}\right] =$$

$$= \left[\begin{array}{c|c} I - M_{2_1 3} M_{3 2_1} & M_{2_1 2_2} - M_{2_1 3} M_{3 2_2} \\ \hline M_{2_2 2_1} - M_{2_2 3} M_{3 2_1} & I - M_{2_2 3} M_{3 2_2} \end{array}\right] \qquad (104)$$

For the sake of simplicity we shall write this as

$$\Omega^{-1} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \qquad (105)$$

Figure 7 shows the form of Ω^{-1}.

7. The pattern of matrix Ω^{-1}

$m_1=5$, $m_3=3$, $v=2$

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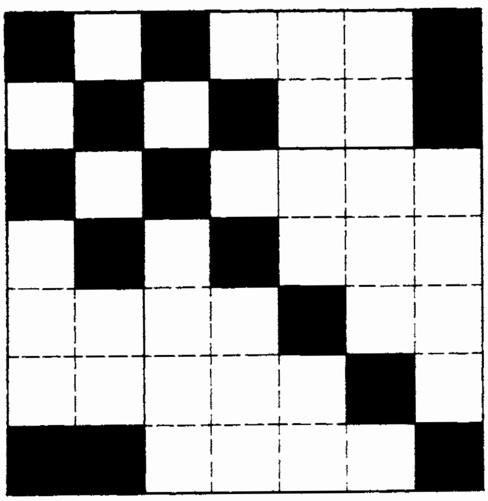

Elements $\theta_{221_1 l_2 h_1 h_2}$ of Θ are identical to elements $w_{l_1 l_2 h_1 h_2}$ mentioned in (98). For the other elements of Θ.

$$
\begin{aligned}
&\theta_{111_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \\
&\theta_{111\ l\ h_1 h_2} = \delta_{h_1 h_2} - g_{lm_2 h_1}\, g_{lm_2 h_2} - g_{m_1 lh_1}\, g_{m_1 lh_2} && \\
&\theta_{121_1 l_2 h_1 h_2} = 0 && l_1 \neq l_2 \ \vee\ l_2 < m_1 \qquad (106) \\
&\theta_{121\ l\ h_1 h_2} = g_{lm_2 h_1}\, g_{lm_2 h_2} && \\
&\theta_{121\ m_1 h_1 h_2} = -\, g_{m_1 lh_1}\, g_{m_1 lh_2} &&
\end{aligned}
$$

To get matrix Ω we must invert Ω^{-1}. For this we find

$$
\Omega = \begin{bmatrix} (\Theta_{11}-\Theta_{12}\Theta_{22}^{-1}\Theta_{21})^{-1} & -\Theta_{22}^{-1}\Theta_{21}(\Theta_{11}-\Theta_{12}\Theta_{22}^{-1}\Theta_{21})^{-1} \\ -(\Theta_{11}-\Theta_{12}\Theta_{22}\Theta_{21})^{-1}\Theta_{12}\Theta_{22}^{-1} & \Theta_{22}^{-1}+\Theta_{22}\Theta_{21}^{-1}(\Theta_{11}^{-1}-\Theta_{12}\Theta_{22}^{-1}\Theta_{21})^{-1}\Theta_{12}\Theta_{22}^{-1} \end{bmatrix} \qquad (107)
$$

There are two inverses in (107). Inverse $\Theta_{22}{}^{-1}$ must be computed from the block diagonal matrix and thereby becomes a block diagonal matrix itself. The inverse of the block diagonal matrix can be computed by inverting each m_1 symmetrical block with dimensions m_3 x m_3 separately. The other inverse that has to be computed in (107) is a completely filled symmetrical matrix with dimensions (v x m_3) x (v x m_3).

Here, too, the described method requires much more calulating time than the RAS method. However, it does have three advantages: 1. The adjustment takes plaᴗe while the uncertainties with which the cells are calculated are taken into account; 2. uncertainties are permitted in the row and column sums; 3. for pairs of row and column sums that are required to be identical, two independent (different) estimates with different uncertainties can be given. The required number of operaations is approximately $[\ (m_2 \times m_3)^3 + (v \times m_3)^3 + m_1\, m_3{}^3\ ] / 6$.

12. Higher-dimensional tables

In the preceding chapters we have seen how to adjust two and three-dimensional tables to row and column sums, whether or not estimated with uncertainties, whereby additional rstraints can also occur. We may now wonder whether higher-dimensional tables can also be dealt with in this way. The method generally requires the calculation of sets of multipliers. The space in which these sets are defined (a hyper-plane) have one dimension less than the space of the set of cells in the table. This means that for a four-dimensional table, three-dimensional sets of multipliers will have to be determined. A table of any realistic proportions would thus require the computation of an unrealistically large number of multipliers; for this reason, we shall not developt a method for such a case here.

For the three-dimensional tables we assumed that the sum across one dimension was given by the table for every element of a side plane. It would of course be possible that instead of this, for one or more side planes the table gives the sum across two dimensions for every element of one rib. The method can be worked out for each of the three possible situations (one rib and two planes, two ribs and one plane, three ribs). Obviously this also applies for higher-dimensional tables with sums across the hyper-ribs; this is, however, only meaningful if the sought set of multipliers has a maximum two dimensions. Such situations are all so specific that they should only be worked out when they occur.

Conclusion

For the adjustment of two-dimensional tables the Lagrange method is by far preferred above the usually applied RAS method. Besides the advantage of the possibility to take into account the confidence intervals of the individual elements the Lagrange method is in general far more economic in the need for computer capacity.

For three-dimensional tables the Lagrange method will in general demand more computing time than the RAS method. However, in the case of long thin tables ("shelves") the opposite may be the case. Nevertheless, the application of the Lagrange method is still sometimes preferred if there exist additional constraints.

The adjustment of higher-dimensional tables with the aid of Lagrange multipliers will in general require too much computing time for present day computers. Exceptions may be found for very small higer-dimensional tables.

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Available National Accounts Occasional Papers

NA/01 *Flexibility in the system of National Accounts*, Eck, R. van, C.N. Gorter and H.K. van Tuinen (1983)
This paper sets out some of the main ideas of what gradually developed into the Dutch view on the fourth revision of the SNA. In particular it focuses on the validity and even desirability of the inclusion of a number of carefully chosen alternative definitions the "Blue Book", and the organization of a flexible system starting from a core that is easier to understand than the 1968 SNA.

NA/02 *The unobserved economy and the National Accounts in the Netherlands, a sensitivity analysis*, Broesterhuizen, G.A.A.M. (1983)
This paper studies the influence of fraud on macro-economic statistics, especially GDP. The term "fraud" is used as meaning unreporting or underreporting income (e.g. to the tax authorities). The conclusion of the analysis of growth figures is that a bias in the growth of GDP of more than 0.5% is very unlikely.

NA/03 *Secondary activities and the National Accounts: Aspects of the Dutch measurement practice and its effects on the unofficial economy*, Eck, R. van (1985)
In the process of estimating national product and other variables in the National Accounts a number of methods is used to obtain initial estimates for each economic activity. These methods are described and for each method various possibilities for distortion are considered.

NA/04 *Comparability of input-output tables in time*, Al, P.G. and G.A.A.M. Broesterhuizen (1985)
In this paper it is argued that the comparability in time of statistics, and input-output tables in particular, can be filled in in various ways. The way in which it is filled depends on the structure and object of the statistics concerned. In this respect it is important to differentiate between coordinated input-output tables, in which groups of units (industries) are divided into rows and columns, and analytical input-output tables, in which the rows and colunms refer to homogeneous activities.

NA/05 *The use of chain indices for deflating the National Accounts*, Al, P.G., B.M. Balk, S. de Boer and G.P. den Bakker (1985)
This paper is devoted to the problem of deflating National Accounts and input-output tables. This problem is approached from the theoretical as well as from the practical side. Although the theoretical argument favors the use of chained Vartia-I indices, the current practice of compilating National Accounts restricts to using chained Paasche and Laspeyres indices. Various possible objections to the use of chained indices are discussed and rejected.

NA/06 *Revision of the system of National Accounts: the case for flexibility*, Bochove, C.A. van and H.K. van Tuinen (1985)
This paper examines the purposes of the SNA and concludes that they frequently conflict with one another. Consequently, the structure of the SNA should be made more flexible. This can be achieved by means of a system of a general purpose core supplemented with special modules. This core is a full-fledged, detailed system of National Accounts with a greater institutional content than the present SNA and a more elaborate description of the economy at the meso-level. The modules are more analytic and reflect special purposes and specific theoretical views. It is argued that future revisions will concentrate on the modules and that the core is more durable than systems like present SNA.

NA/07 *Integration of input-output tables and sector accounts; a possible solution*, Bos, C. v.d. (1985)
In this paper, the establishment-enterprise or company problem is tackled by taking the institutional sectors to which the establishments belong into account during the construction of input-output tables. The extra burden on the construction of input-output tables resulting from this approach is examined for the Dutch situation. An adapted sectoring of institutional units is proposed for the construction of input-output tables. The proposed approach contains perspectives on further specification of the institutional sectors,

households and non-financial enterprises and quasi-corporate enterprises.

NA/08 *A note on Dutch National Accounting data 1900-1984*, Bochove, C.A. van (1985)
This note provides a brief survey of Dutch national accounting data for 1900-1984, concentrating on national income. It indicates where these data can be found and what the major discontinuities are. The note concludes that estimates of the level of national income may contain inaccuracies; that its growth rate is measured accurately for the period since 1948; and that the real income growth rate series for 1900-1984 may contain a systematic bias.

NA/09 *The structure of the next SNA: review of the basic options*, Bochove, C.A. van and A.M. Bloem (1985)
There are two basic issues with respect to the structure of the next version the UN System of National Accounts. The first is its 'size ': reviewing this issue, it can be concluded that the next SNA must be 'large ' in the sense of containing an integrated meso-economic statistical system. It is essential that the next SNA contains an institutional system without the imputations and attributions that pollute present SNA. This can be achieved by distinguishing, in the central system of the next SNA, a core (the institutional system), a standard module for non-market production and a standard module describing attributed income and consumption of the household sector.

NA/10 *Dual sectoring in National Accounts*, Al, P.G. (1985)
The economic process consists of various sub-processes, each requiring its own characteristic classification when described from a statistical point of view. In doing this, the interfaces linking the sub-systems describing the individual processes must be charted in order to reflect the relations existing within the overall process. In this paper, this issue is examined with the special refernce to dual sectoring in systems of National Accounts. Following a conceptual explanation of dual sectoring, an outline is given of a statistical system with complete dual sectoring in which the linkages are also defined and worked out. It is shown that the SNA 1968 is incomplete and obscure with respect to the links between the two sub-processes.

NA/11 *Backward and forward linkages with an application to the Dutch agro-industrial complex*, Harthoorn, R. (1985)
Some industries induce production in other industries. An elegant method is developed for calculating forward and backward linkages avoiding double counting. For 1981 these methods have been applied to determine the influence of Dutch agriculture in the Dutch economy in terms of value added and labour force.

NA/12 *Production chains*, Harthoorn, R. (1986)
This paper introduces the notion of production cains as a measure of the hierarchy of industries in the production process. Production chains are sequences of transformation of products by successive industries. It is possible to calculate forward transformations as well as backward ones.

NA/13 *The simultaneous compilation of current price and deflated input-output tables*, Boer, S. de and G.A.A.M. Broesterhuizen (1986)
This paper discusses a number of aspects of the procedure according to which input-output tables are compiled in the Netherlands. A few years ago this method underwent an essential revision. The most significant improvement means that during the entire statistical process, from the processsing and analysis of the basic data up to and including the phase of balancing the tables, data in current prices and deflated data are obtained simultaneously and in consistency with each other. Data in current prices first used to be compiled and data in constant prices and changes in volume and prices used to be estimated only afterwards. With the new method the opportunity for the analysis of the interrelations between various kinds of data, and thus better estimates is used.

NA/14 *A proposal for the synoptic structure of the next SNA*, Al, P.G. and C.A. van Bochove (1986)

NA/15 *Features of the hidden economy in the Netherlands*, Eck, R. van and B. Kazemier (1986)
This paper presents survey results on the size and structure of the hidden labour market in the Netherlands.

NA/16 *Uncovering hidden income distributions: the Dutch approach*, Bochove, C.A. van (1987)

NA/17 *Main national accounting series 1900-1986*, Bochove, C.A. van and T.A. Huitker (1987)
The main national accounting series for the Netherlands, 1900-1986, are provided, along with a brief explanation.

NA/18 *The Dutch economy, 1921-1939 and 1969-1985. A comparison based on revised macro-economic data for the interwar period*, Bakker, G.P. den, T.A. Huitker and C.A. van Bochove (1987)

NA/19 *Constant wealth national income: accounting for war damage with an application to the Netherlands,1940-1945*,Bochove, C.A. van and W. van Sorge (1987)

NA/20 *The micro-meso-macro linkage for business in an SNA-compatible system of economic statistics*, Bochove, C.A. van (1987)

NA/21 *Micro-macro link for government*, Bloem, A.M. (1987)
This paper describes the way the link between the statistics on government finance and national accounts is provided for in the Dutch government finance statistics.

NA/22 *Some extensions of the static open Leontief model*, Harthoorn, R. (1987)
The results of input-output analysis are invariant for a transformation of the system of units. Such transformation can be used to derive the Leontief price model, for forecasting input-output tables and for the calculation of cumulative factor costs. Finally the series expansion of the Leontief inverse is used to describe how certain economic processes are spread out over time.

NA/23 *Compilation of household sector accounts in the Netherlands National Accounts*, Laan, P. van der (1987)
This paper provides a concise description of the way in which household sector accounts are compiled within the Netherlands National Accounts. Special attention is paid to differences with the recommendations in the United Nations System of National Accounts (SNA).

NA/24 *On the adjustment of tables with Lagrange multipliers*, Harthoorn, R. and J. van Dalen (1987)
An efficient variant of the Lagrange method is given, which uses no more computer time and central memory then the widely used RAS method. Also some special cases are discussed: the adjustment of row sums and column sums, additional restraints, mutual connections between tables and three dimensional tables.

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