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SOME EXTENSIONS OF THE STATIC OPEN LEONTIEF MODEL

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Nr. NA-022
1987

Summary

The values in a homogeneous input-ouput table can be seen as a measure for the quantities of products involved. This implies that these values lay down a specific system of units. This paper demonstrates that the results of input-output analysis are invariant for a transformation of the system of units. By choosing a special transformation we can derive the Leontief price model. The gauge transformation is defined as a transformation in which the values of the table elements are gauged according to the addition per industry; in the case of added value formation this results in a table in current prices. These techniques can be used for forecasting IO-tables. The forecast should be made on the basis of the forecast final demand, where necessary in combination with restrictions on total production. In doing this, the selection vector technique is a useful tool. These selection vectors are also important for the calculation of cumulative facet costs. By extending the IO-table with price indices for the consumption by the households it becomes possible to study the phenomenon of inflation. It is demonstrated that in a simple model, the wage-price spiral converges. Under certain circumstances, the series expansion of the Leontief inverse offers the possiblity of describing how certain economic processes are spread out over time.

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1. Introduction

When it produces certain commodities (here the term commodities includes the combination of goods and services), a company generally uses commodities produced by other companies. In addition, these companies also incur so-called primary costs. If we assume that a company wastes nothing, the use of commodities is a physical necessity for the realization of production. Whether a company actually starts production will depend to a great extent on whether they can expect to sell their commodities (in the long run) for a sufficient price. This 'sufficient' price is expressed in the operating surplus, a component of the company's primary costs. Here we argue that primary costs play a part comparable with that of the consumption of commodities: the primary costs can be considered as (abstract) goods. In this point of view the value of the production is determined completely by consumption.

The consumption of commodities by final and intermediary users and the production by companies can be entered in a matrix: the input-output table (IO-table). However, such a table would take on gigantic proportions if all the commodities were treated separately; in practice, therefore, companies and commodities are dealt with on a group-by-group basis in order to construct an IO-table with reasonable proportions. However, this does lead to so-called heterogeneity problems: the commodities or companies taken together are not always comparable with each other. A practical solution to this problem can be found in expressing the commodities flows in the IO-tables in monetary units, although these monetary units can be seen as representing physical amounts. In the process of drawing up IO-tables for 1919 and 1929, Leontief [1], for example, said: 'All figures indicate millions of dollars. In order to obtain the corresponding physical amounts of all commodities and services, we simply define the unit of physical measurement of every particular type of product so as to make it equal to that amount of the commodity which can be purchased for one dollar at prevailing prices'.

The continued use of IO-tables in values has meant that the idea of physical amounts has become relegated to the background. The ease with which very different units can be added together (often through sheer necessity) when values are used has been partly to blame for this. This has probably also led to the fact that a favourable aspect of an IO-table in values (current prices) is often wrongly imposed on other IO-tables. This favourable aspect is the equality of column and row totals for each industry, which in turn is the consequence of the accountant's demand that every company's total expenditure equals its total receipts.

The property of identical row and column totals measured in values is a fine tool for constructing IO-tables in current prices, though it is not prerequisite for the possible applications of the table. The application of a table containing elements in physical amounts, for which this condition does not hold, since the amounts of the various commodities cannot be added together, will usually give the same results. Here we shall show that the results of input-output analysis are invariant for transformation of the unit system. We shall call the transformation of an arbitrary system of units to a system which does justice to the relative value (in economic trafic or in another sense) of the commodities a gauge transformation. In the gauge transformation the gauge is the result of the definition of a common scale. A table in current prices is the result of a gauging of the individual exchange values in economic traffic.

We shall also demonstrate that the construction of IO-tables in constant prices can be seen as a transformation on the system of units of a table in current prices. As this is not usually a gauge transformation, a table in constant prices need not have identical row and column totals, unless - as is quite common at statistical bureaus - the deflated value added (or operating surplus) is defined as the differnce between deflated production and deflated consumption.

Forecasting IO-tables with the aid of a scenario will also be looked into. The gauge is put aside here too so that the demand for identical row and column totals no longer applies. However, if all the necessary information (or relevant estimates) is thought to be available, it is possible to construct a gauge transformation. As accurate estimates of price trends are not usually possible, the application of such a gauge transformation will lead to a rigid determination of price effects of technological changes or certain assumed price changes (e.g. of raw materials or primary cost categories). We call this a rigid determination, because other more flexible effects on the prices as for instance the flexible market mechanism are not taken into account.

An often used form of input-output analysis is the determination of the so-called facet requirements and cumulative facet costs. In the IO-table a facet may sometimes take the form of pseudo industry with its own input structure. In the determination of cumulative facet costs in particular this will lead to problems; problems easily solved with the aid of selection vectors. This is explained with the aid of an example from energy analysis.

Finally, we examine the application of IO-tables in determining expected inflation and the occurrence of time lags. To determine inflation as a consequence of initial price rises, the IO-table is extended with relations other than the usual production-consumption links. These new relations behave as (pseudo) industries. Time lags, e.g. those related to inflation, can be determined by using the series expansion in the Leontief inverse. As the lags between every pair of industries may differ, the IO-table should be modified in a simple way to be able to represent these pair-by-pair delays.

2. The IO-table

Consumption of commodities by various companies and the purchase of commodities by final expenditure categories can be entered in a matrix; this gives us a so-called commodities x companies table. In the same way, the production of commodities by various companies can also be described in a matrix: we then get a companies x commodities table. For any national economy on a realistic scale, such tables are unmanageably large. One obvious solution towards manageable proportions is to approach companies and commodities as groups, taking companies that show a strong similarity together to form an industry; for the commodities a slightly different solution is usually opted for. First of all commodities of a similar form are divided according to the group of companies (industry) that produce them; then the various commodities produced by the same industry are taken together. In this way we get a matrix with production and consumption by industries: a so-called industries x industries table. This is the most common form of IO-table.

In the first two tables mentioned (commodities x companies and companies x commodities) the commodity flows can be entered in a natural way in physical amounts. This does not pose any problems for the combination of companies into industries. Combining various sorts of commodities, however, does pose problems. Every sort of commodity has a different dimension. The amount of commodities must be recorded in a unit corresponding to the dimension concerned; e.g. kg of commodity 1, kg of commodity 2, litre of commodity 3, pieces of commodity 4 etc. which correspond respectively with the dimensions weight of commodity 1, weight of commodity 2, volume of commodity 3, numbers of commodity 4 etc. Combining commodities with different units is in principle impossible. This is usually solved by taking not the physical amounts of the commodities concerned, but their value in economic traffic. The idea behind this is that the various commodities as produced by a certain industry have initiated a consumption that is identical as far as relative composition is concerned. According to this train of thought, the price of the products of a certain industry shows that the product represents a package of consumed commodities plus a factor added by that industry group itself. The value of the product is thus usually greater than the value of the consumption. In the way outlined above a so-called industries x industries table was constructed; for, the consumed commodities are combined to groups of commodities produced by one and the same industry.

The above-mentioned IO-table is an idealization of the practical possibilities. It is an ideal homogeneous table: for every part of production by an industry expressed in monetary units there is a corresponding (intermediary or final) consumption, equal as far as relative composition is concerned (in monetary units). Reality is very different: many companies manufacture a wide range of products, each entailing a differently composed consumption. There are also identical products produced by different companies that are assigned to different industries. Also, an identical product can even be produced in entirely different ways from one company to another, resulting in pronounced differences in the consumption by such industries. These heterogeneity problems would lead to insurmountable problems in a table with physical amounts; when monetary units are used, the problems are hardly noticed.

A practical starting point for many input-output studies is an IO-table containing monetary units. Such a table is usually 'polluted' with a certain degree of heterogeneity. In spite of this, it can be considered as isomorphous with a table containing amounts, in which the observed amounts have a (hidden) heterogeneity 'interference'. Here, each monetary unit of product of a certain industry represents an equal combination in physical quantities of products. All the products are now recorded in the same unit; but the dimensions, and to a certain extent the units as well, are still different. The different dimensions in such a table are value of commodity 1, value of commodity 2 etc. The isomorphism with the dimensions in the case of registration in physical amounts is evident.

Further on we shall see that the equality of the units is the consequence of a gauge transformation. Although this equality sometimes leads to very useful conclusions, it can also be confusing, in the same way that expressing the speed and the weight of a car in energy units is confusing: speed leads to kinetic energy, weight can be seen as the equivalent of mass which - according to the relativity theory - is identical to energy. Such a system could be very valuable in theoretical physics, but not in everyday life. A similar problem occurs in input-output analysis: the value flows can supply very useful information, but if we want to analyse production processes in their physical relations, we must keep the idea of their representing physical flows in our mind.

3. Invariance

The structure of the relevant part of the IO-table is given by figure 1. The submatrix of intermediary consumption is square shaped as we start out

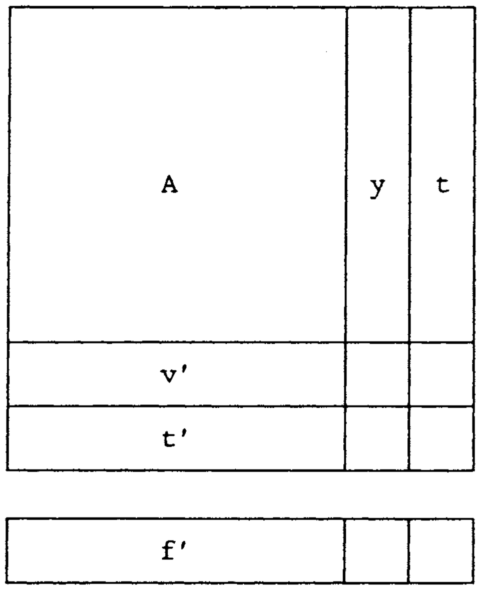

Figure 1. The structure of the input-output table.
A = intermediary consumption;
y = final demand;
t = total production (t' = total consumption);
v' = primary input;
f' = facet input.

from an industries x industries table. We imagine the final demand and primary costs as a column and a row vector respectively; the further division is a complication which is not relevant here. For a table in current prices, the row totals t equal the corresponding column totals t'. Any facet (e.g. energy consumption) that is related to the economic behaviour by intermediary and final consumers can be recorded in a vector. Such a vector can be seen as an 'appendix' to the IO-table, more or less comparable with the primary costs; the natural place for this type of quantities is in a row under the IO-table. A facet is thus a sort of input (even in the case of air pollutants) which usually brings along its own system of units and is not usually expressed in values. Chapter 5 will show that after a so-called gauge transformation, here a facet in an IO-table can play an identical role as the primary costs. The invariance characteristic to be derived below, by the way, still applies if we replace the row vector f' by a matrix F consisting of several (components of) facets (each with its own unit).

From the quantities in the IO-table of figure 1, we can now determine the various coefficients:

intermediary input coefficients: $\tilde{A} = A\ \hat{t}^{-1}$ (1)

primary input coefficients: $\tilde{v}' = v'\ \hat{t}^{-1}$ (2)

facet input coefficients: $\tilde{f}' = f'\ \hat{t}^{-1}$ (3)

where $\hat{t}$ stands for a diagonal matrix with the elements of the corresponding column vector t on the diagonal element of the row of $\hat{t}$.

The matrix of cumulative intermediary input coefficients (the wellknown Leontief inverse) is then given by

$$(I - \tilde{A})^{-1} \quad (4)$$

With the aid of this, the cumulative facet input coefficients can be determined

$$\tilde{f}' (I - \tilde{A})^{-1} \quad (5)$$

and thus also the facet requirements f'_y (i.e. the required cumulative contribution of the facet) corresponding with the total final demand y

$$f'_y = \tilde{f}' \, (I - \tilde{A})^{-1} \, \hat{y} \qquad (6)$$

This equation is the basis for the standard IO-analysis.

As we are not committed to a special choice for the facet, we could take the vector of the primary costs, or part of it such as value added, instead of f. The following chapter, which goes into prices, will make use of this aspect.

Equation (6) must be independent of the measuring standards and thus also of the system of units used in the IO-table. In principle, every row in the IO-table represents a separate kind of quantity; this means that every row has its own dimension, so that all the elements in a row must be written in the same unit, which suits the dimension concerned. Due to their different dimensions, elements of different rows are described with the aid of different units. The independence of (6) of the system of units in combination with the separate unit for every row implies that (6) must be invariant for a change of unit for each row separately. Such a change of the system of units can be described as a premultiplication by a diagonal matrix $\hat{r}$, where $\hat{r}$ is non-singular. From this it follows that (6) must be invariant under a simultaneous transformation of A, y and t by premultiplication by a nonsingular diagonal matrix $\hat{r}$:

$$
\begin{aligned}
f'_y &= f' \, \hat{t}^{-1} \, \{ I - A \, \hat{t}^{-1} \}^{-1} \, \hat{y} \\
&= f' \, (\hat{r} \, t)^{-1} \, \{ I - (\hat{r} \, A) \, (\hat{r} \, t)^{-1} \}^{-1} \, (\hat{r} \, y) \\
&= f' \, \hat{t}_r^{-1} \, \{ I - A_r \, \hat{t}_r^{-1} \}^{-1} \, \hat{y}_r \qquad (7)
\end{aligned}
$$

(by calling the transformed quantities A_r, y_r, and t_r respectively, the result of the simultaneous transformation has exactly the same mathematical form as the original expression). This can easily be proven under the assumption that the inverse matrices $\hat{r}^{-1}$ and $\hat{t}^{-1}$ exist, i.e. both $\hat{r}$ and $\hat{t}$ should be non-singular. The matrix $\hat{t}$ is non-singular if all elements of the vector t are non-zero, while $\hat{r}$ was chosen to be nonsingular. So the proof below is valid for the case that the IO-table contains no industries with zero production.

Proof:

$$f' (\hat{r}\, t)^{-1} \{ I - (\hat{r} A) (\hat{r}\, t)^{-1} \}^{-1} (\hat{r}\, y)$$

$$= f' (\hat{r}\, \hat{t})^{-1} \{ I - \hat{r} A (\hat{r}\, \hat{t})^{-1} \}^{-1} \hat{r}\, \hat{y}$$

$$= f' \hat{t}^{-1} \hat{r}^{-1} \{ I - \hat{r} A \hat{t}^{-1} \hat{r}^{-1} \}^{-1} \hat{r}\, \hat{y}$$

$$= \tilde{f}' \hat{r}^{-1} \{ I - \hat{r} \tilde{A} \hat{r}^{-1} \}^{-1} (\hat{r}^{-1})^{-1} \hat{y}$$

$$= \tilde{f}' \{ \hat{r}^{-1} (I - \hat{r} \tilde{A} \hat{r}^{-1}) \hat{r} \}^{-1} \hat{y}$$

$$= \tilde{f}' \{ \hat{r}^{-1} \hat{r} - \hat{r}^{-1} \hat{r} \tilde{A} \hat{r}^{-1} \hat{r} \}^{-1} \hat{y}$$

$$= \tilde{f}' \{ I - \tilde{A} \}^{-1} \hat{y}$$

$$= f'_y \qquad \text{q.e.d.}$$

4. Prices

In one of his earlier studies, Leontief [1] had already posed that his IO-tables containing values could be considered as tables with comparable physical amounts. This was founded on the assumption of absolute homogeneity: every unit value of the production recorded on one and the same row represents the same amount of an identical product. In practice this absolute homogeneity is obviously not relaized; the heterogeneity introduced need not necessarily be a fundamental error in the table, but can be seen as a form of interference.

A table in current prices has the following 'favourable' characteristic:

the corresponding row and column totals are identical:

$$A\,i + y = t \tag{8a}$$

$$i'\,A + v' = t' \tag{8b}$$

where i stands for the so-called summation vector of which all the elements have the value one. From (8b) we get by postmultiplication by $\hat{t}^{-1}$

$$i'\,\tilde{A} + \tilde{v}' = i' \tag{9}$$

which can also be written in the form that we will call the unit theorem

$$\tilde{v}'\,(I - \tilde{A})^{-1} = i' \tag{10a}$$

The interpretation of (10a) is that every unit value can be conceived as a cumulation of primary costs.

Equation (10a) can be rewritten in a more complicated manner

$$\tilde{v}'\,(I - \tilde{A})^{-1}\,\hat{j} = i' \tag{10b}$$

where j equals the summation vector i (so $\hat{j}$ equals the unit matrix I). This more complicated way of writing (10a), where a different symbol is used for the identical vectors i and j, means that (10b) is similar to (6). The vector of the value added which is present in the IO-table has now taken on the character of a facet. This means that for (10b) the same invariance characteristics apply: the expression in (10b) is invariant under a simultaneous premultiplication of A, t and j with any non-singular diagonal matrix $\hat{r}$:

$$\tilde{v}' (I - \tilde{A})^{-1} \hat{j} = v' \hat{t}^{-1} (I - A \hat{t}^{-1})^{-1} \hat{j}$$

$$= v' (\hat{r} t)^{-1} (I - (\hat{r} A) (\hat{r} t)^{-1})^{-1} (\hat{r} j)$$

$$= i' \tag{11}$$

We were free to choose any non-singular diagonal matrix $\hat{r}$ in (11) so this relation must still also hold if we choose a special non-singular $\hat{r}$. If we choose $\hat{r}$ in such a way that it represents the relative prices at time τ in relation to time 0:

$$\hat{r} = \hat{p}_\tau \hat{p}_0^{-1} \tag{12}$$

where p_τ is prices at time τ and p_0 those at time 0. Here the identity (10b) applies for time 0. From 11 it then follows:

$$i' = \tilde{v}' (I - \tilde{A}_0)^{-1} \hat{j}$$

$$= v' \hat{t}_0^{-1} (I - A_0 \hat{t}_0^{-1})^{-1} \hat{j}$$

$$= v' (\hat{r} \hat{t}_0)^{-1} (I - \hat{r} A_0 (\hat{r} \hat{t}_0)^{-1})^{-1} \hat{r} \hat{j}$$

$$= v' \hat{t}_\tau^{-1} (I - A_\tau \hat{t}_\tau^{-1})^{-1} \hat{r} \tag{13}$$

where the indices refer to quantities valued in prices at time 0 or τ. From the result of (13) it directly follows that

$$v' \hat{t}_\tau^{-1} (I - \tilde{A}_\tau)^{-1} = i' \hat{r}^{-1} \tag{14}$$

The expression in (14) therefore supplies the reciprocals of the price indices. By way of the invariance characteristics of the Leontief inverse, the classic Leontief price model [2] is again derived.

With the aid of (14) it is in principle possible to convert the primary prices found for time τ to primary costs in prices of a base year 0 (time τ can be either before or after time 0) in aid of an IO-table in constant prices. This will then lead to a table in which the column totals equal the corresponding row totals (see e.g. next chapter). However, it is very doubtful whether we can call this a deflation of the primary costs. If the primary costs are composed of one component, which can be deflated easily, such as labour, then (14) will indeed give the right deflator. If the primary costs are built up of several easily deflated components, (14) will give the right deflators too (then the vector v is the result of the summation of the components of a matrix V of primary costs; every element of v now has its own deflator as it is built up of a combination of components which each have their own deflator). The deflator is only correct if all price changes are a consequence of a price change of the primary costs. This means that the consumption amounts of both intermediary goods and the primary costs remain the same for each unit of product; in other words, the production structure does not respond to price changes. The problem is, however, that trends in time do not only entail price trends but also trends in amounts of primary costs. With the aid of (14), these amount trends can be traced only in combination with the price trends.

If we study primary costs more closely, we find components such as imports, depreciation and labour costs, in which the concept of price poses no great problems. There are also components such as other income, taxes, subsidies (including labour costs and, sometimes, depreciations, we call all these components together in short value added), for which the concept of price is not so clear. By means of a special trick these last 'intangible' components can be given a physical form which can in turn be assigned a price (here we give an illustration without claiming that this is <u>the</u> right way of deflating). Suppose it became mandatory to pay taxes or transfer other income in none other than gold bullion. At that moment the intangible components turn into tangible ones with 'weight in gold' as their dimension. It will be obvious that in this way a (primary) input in gold is required which certainly does have a price tag. Its price can, just like other prices, fluctuate in time; in constructing an IO-table in constant prices the value added must then also be deflated. The value added does not distinguish itself from other inputs of industry groups in this. By deflating the (components of the) value added by a fixed factor, the column totals in a table in constant prices will no longer equal the corresponding row totals.

In the IO-table a price change is manifested as a multiplication by a factor for each (homogeneous) row. This means that the switch from one sort of prices to another is equivalent to a change in the system of units in relation to the physical amounts. The method which is sometimes applied in deflating IO-tables (constructing IO-tables in constant prices), altering the value added for each industry group separately, implies that the value added of the individual industries are seen as incomparable quantities.

There are all sorts of practical objections against the abovementioned method. For an actual deflation, for example, a more suitable deflator than gold should be chosen. As in practice the consumer price index is often used as a measure for the devaluation of money, this might be a good candidate for a deflator. But other price indices too (e.g. those for general consumption or capital goods) are worth considering. Choosing an index as a deflator means that something implicit is stated about the way income is spent, while this has nothing to do with the production process. This introduction of an income expenditure element in the description of the production structure is seen by statistical offices as an fundamental objection for the deflation of the operating surplus (or value added). For certain analytical applications, it obviously need not be an objection and may even give extra insight.

In consequence of this fundamental objection, many statistical bureaus have decided to define the operating surplus in a table with constant prices in the same way as in a table with current prices: i.e. as the difference between the values of total production and total consumption by an industry. This is not a deflation in the above sense, however; in practice, it is sometimes called double deflation.

If the operating surplus is deflated directly, profits will be shown in profits and losses in losses. If the operating surplus is defined as the difference between production and consumption (double deflation) this need not necessarily be so: a positive balance in current prices may turn into a negative balance in constant prices. Such a development is often indicative of more efficient production, but this may also sometimes be teh reflection of exogenous imposed prices. Generally speaking, the development in time of the operating surplus defined as the difference between production and consumption of one certain year indicates how the terms of trade for the various industries change. The possible switch switch from profit to loss is adrawback for many analytical applications of IO-tables.

The deflation method described above including the 'deflation' of the value added can also be used the other way around. Starting out from a certain production structure and knowledge of changes that are to occur therein, a new price system can be calculated for a future point in time. For this change in production structure, a change in the amount of created value added is seen as a component of the change in production structure. In this way, changes are introduced in the various price relations rigidly and systematically. The systematical method excludes the important market mechanism. However, it can also be assumed that there is prior knowledge about the effects of the flexible market mechanism, and that these effects are included implicitly by means of the formation of value added in the change of the 'amount' of value added.

In constructing an IO-table according to a certain scenario for economic developments, starting out from a table in either constant or current prices, the elements can be simply multiplied by the factors by which the inputs - measured in amounts - are expected to change. With this table new shadow prices are calculated according to the Leontief price model. With the aid of this, the IO-table can be assembled in values. The latter table represents the relations of the values between the various cells as expected for the production structure concerned. The values that can really be expected are calculated by multiplying all the elements in the table, but this time by the same factor for all the elements. In a certain sense, this last factor represents the general inflation of money. A similar factor will be encountered, by the way, if the table with values is converted from one currency to another.

5. Gauge transformation

Suppose we have an IO-table in some system of units; we can single out variables A and t. In addition a certain facet f is given as a multiple of a certain appropriate unit. From the invariance of (7) it follows that we can determine the cumulative contribution s of the facet f in the total output t.

$$s' = f' \hat{t}^{-1} \{ I - A \hat{t}^{-1} \}^{-1} \hat{t} \tag{15}$$

We now call the simultaneous transformation of A and t (and where applicable y) by premultiplication by the diagonal matrix $\hat{s}$ the gauge transformation. We call this a gauge transformation because the following now applies:

$$f' \hat{s}^{-1} + i' \hat{s} \hat{t}^{-1} A \hat{s}^{-1} = i' \tag{16}$$

The system of units has changed to such an extent that the unit theorem is again applicable.

Proof:

$$f' \hat{s}^{-1} + i' \hat{s} \hat{t}^{-1} A \hat{s}^{-1}$$

$$= [f' + i' \hat{s} \hat{t}^{-1} A] \hat{s}^{-1}$$

$$= [f' + s' \hat{t}^{-1} A] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} (I - A \hat{t}^{-1})^{-1} \hat{t} \hat{t}^{-1} A] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} (I - \tilde{A})^{-1} A] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} (I - \tilde{A})^{-1} \tilde{A} \hat{t}] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} \{ (I - \tilde{A})^{-1} - I \} \hat{t}] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} (I - \tilde{A})^{-1} \hat{t} - f' \hat{t}^{-1} \hat{t}] \hat{s}^{-1}$$

$$= [f' + f' \hat{t}^{-1} (I - A \hat{t}^{-1})^{-1} \hat{t} - f'] \hat{s}^{-1}$$

$$= [f' \hat{t}^{-1} (I - A \hat{t}^{-1})^{-1} \hat{t}] \hat{s}^{-1}$$

$$= s' \hat{s}^{-1}$$

$$= i' \qquad \text{q.e.d.}$$

The consequence of the gauge transformation is that the following relation applies:

$$(f' + i' \hat{s} \hat{t}^{-1} A)^T = \hat{s} \hat{t}^{-1} A i + \hat{s} \hat{t}^{-1} y$$

$$= s \tag{17}$$

(the symbol T indicates a transposition) which is equivalent to the statement that the column totals of the transformed table are identical to the row totals. From equation (16) it follows directly that the left hand side of (17) equals s. For the first line of the right hand side we find

$$\hat{s}\,\hat{t}^{-1} A\, i + \hat{s}\,\hat{t}^{-1} y = \hat{s}\,\hat{t}^{-1} (A\, i + y)$$

$$= \hat{s}\,\hat{t}^{-1}\, t$$

$$= \hat{s}\, i$$

$$= s \qquad\qquad \text{q.e.d.}$$

From (17) it can be derived that the result of two gauge transformations in succession is identical to one direct gauge transformation. For, according to (17) following the first gauge transformation the vector with the new row totals equals s, so that for the second gauge transformation, where r replaces s in (15), t should be replaced by s. Now the following applies:

$$(\hat{r}\,\hat{s}^{-1})\, (\hat{s}\,\hat{t}^{-1}) = \hat{r}\,\hat{t}^{-1} \tag{18}$$

The expression in (15) has the same form as (7). This implies that (15) is invariant under premultiplication of A and t by a diagonal matrix, and thus invariant under the gauge transformation. This means that in the case of two successive gauge transformations on the basis of the same facet f, the vector r equals s, thus reducing the second gauge transformation to a premultiplication by the unit matrix.

An example of gauge transformation which reduces to an identity is the gauge of a table in current prices for the phenomenon of the formation of value added. In this case (15) turns into

$$s' = \tilde{v}' \, (I - \tilde{A})^{-1}\, \hat{t} \tag{19}$$

and with the aid of (10a) we now find

$$\hat{s}\,\hat{t}^{-1} = (i'\,\hat{t}\,)\,\hat{t}^{-1} = \hat{t}\,\hat{t}^{-1} = I$$

$$= \hat{t}\,\hat{t}^{-1}$$

$$= I \tag{20}$$

The usual IO-table in current prices now turns out to have been gauged for the formation of value added.

In addition to gauging to the formation of value added, facets like energy consumption, labour input or environmental pollution could be used for gauging. An IO-table obtained in this way with the intermediary matrix A_f:

$$A_f = \hat{s}\,\hat{t}^{-1}\,A_v$$

$$= \hat{s}\,\hat{t}^{-1}\,A \tag{21}$$

portrays the 'flows' of the facet. The matrix A_v is identical to A, the index v is in recognition of the fact that in the usual IO-table 'flows' of formed value added are visualized. Analogously with value added, the direct contribution of a facet could be called energy added, labour added or environmental pollution added respectively. These tables could be used for, among other things, the information theory of Theil [3], if one wanted to study the information content of an IO-table in relation to a certain facet.

6. Complete Leontief inverse

In gauging the intermediary part of an IO-table to a certain facet, the intermediary part is expressed in the same units as the facet concerned. This suggests that in principle there is no difference between the facet used for gauging and the intermediary part of the IO-table. The same sort of thing applies for final demand, which is expressed in the same units as intermediary consumption and primary costs respectively. In view of the similarity between the units of the various submatrices of the IO-table, the next obvious step is to determine the Leontief inverse for the complete IO-table, in other words a matrix in which all the components of the IO-table occur. However, this does require a certain rearrangement of the submatrices; we construct the complete IO-table as follows:

$$C = \begin{bmatrix} A & 0 & Y_A \\ P & 0 & Y_P \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

Submatrices A, P, Y_A and Y_P stand for intermediary consumption, primary costs, final demand for commodities and final demand which is also primary costs respectively. Here the introduction of zero matrices is necessary as the traditional (space-saving) arrangement in the IO-table suggests a <u>direct</u> mutual dependency between final demand and primary costs. The IO-table is not usually square shaped; matrix C is by definition square shaped.

For matrix C we get the input coefficients $\tilde{C}$ by dividing the columns by the corresponding row totals; the columns of final demand are divided by the corresponding column totals as the corresponding row totals equal zero (the original division is not permitted here). The Leontief inverse now becomes

$$(I - \tilde{C})^{-1} = \begin{bmatrix} (I - \tilde{A})^{-1} & 0 & (I - \tilde{A})^{-1} \tilde{Y}_A \\ \tilde{P} (I - \tilde{A})^{-1} & I & \tilde{P} (I - \tilde{A})^{-1} \tilde{Y}_A + \tilde{Y}_P \\ 0 & 0 & I \end{bmatrix} \tag{23}$$

On the first row we find the cumulative input coefficients (Leontief inverse) and the cumulative consumption in aid of the final demand. On the second row we find the cumulative primary cost coefficients and the cumulative primary costs in aid of final demand. In the components of the complete Leontief inverse, we therefore recognise various terms familiar from input-output analysis. The usual Leontief inverse occurs four times and functions here as a sort of cumulation operator. The primary costs can all be found on their own (second) row; the final demand can only be found in its own (last) column. The fact that primary costs are indeed 'primary' is evident from the emptiness of the 'column of primary costs'; apart from the unit matrix on the corresponding row (which therefore refers to itself), primary costs have no 'inputs'. Analogously, final demand contributes nothing to other submatrices as it only occurs in the last column; it is indeed final.

In composing the complete IO-table we took the components of the traditional IO-table for granted as our starting point. This traditional IO-table is gauged to the formation of value added or in the above form to primary costs. In the previous chapter, we saw that a gauging to a completely different facet is very well possible. In that case, this facet takes the place of the primary costs. This equality between value added and other facets suggests the inclusion of those facets in a row of the matrix P (therefore also Y_p). The gauging has then taken place to a component or combination of components (recorded in the same unit) of P. In (23) we then will find the facet requirements by final demand in the second row of the last column.

If more than one facet is included simultaneously in P, no (indirect) dependency between the various facets turns out to exist. This is immediately evident from the form of (23). Due to the lack of dependency between the various facets, a common gauge is in general impossible: the gauge matrix becomes singular.

The construction of the C matrix in (22) means the fundamental difference between the primary and the intermediary part is cancelled. This leads to the conclusion that gauging could also take place to a certain row in the intermediary part, i.e. to an industry. The result would then be a table in which the 'flows' of a certain product are made visible. This may be interesting if the industry concerned is a bottleneck in the economy; e.g. due to capacity or enviromental-technical reasons. Electricity production would be an interesting industry for such a 'flow' table.

It is relatively easy to obtain earlier derived expressions with the aid of the Leontief inverse. The unit theorem, for example, in (10a) can be obtained by

$$j' (I - \tilde{C})^{-1} = i' \tag{24}$$

where j is a vector comprising ones on the rows of the primary costs and otherwise zeros. Vector i is obviously longer here than in (10a).

7. Forecasting IO-tables

Studies are regularly encountered in which an IO-table for a future point in time is estimated as follows. First of all the total production and, where applicable, the components of final demand, per industry is estimated. The intermediary consumption per industry is calculated with the aid of total production, taking into account expected technological developments. Then an estimate is made of the price trends and this is applied to the elements of the table. Finally, it turns out that the column totals do not equal the corresponding row totals and these differences are cancelled with the aid of the so-called RAS method.

The method outlined above contains three independent estimates (total production, technological changes anf price trends) and a smoothing method (RAS method). The first two estimates are inevitable, the estimate of the price trend can be avoided to a large extent while the RAS method is completely superfluous. The latter is particularly important as the production structure, which has been estimated carefully on basis of the best information available, is changed relatively randomly; this change bears no relation at all to the factual production relations to be expected.

A more accurate procedure would be:

1. First of all estimate the final demand or the volume of total

production, for each industry in constant prices.
2. Estimate the consequences of technological changes on the input coefficients in constant prices.
3. From these two, calculate intermediary consumption in constant prices.
4. Calculate the relative prices (shadow prices) according to the Leontief price model.

In this way we get the 'best' estimate of an IO-table in mutually comparable prices. The row totals and the column totals are identical due to the gauge transformation involved in calculating shadow prices, thus a smoothing method like the RAS method becomes superfluous.

The first step is the estimation of final demand or total production volume in constant prices. These are two separate matters. For an important part of production it can be said that the volume, taking into account a certain import volume, is determined by final demand. The cumulative intermediary demand obviously links the various industries. However, part of the production volume is not determined by final demand but by a maximum production capacity in certain industries (the difference between domestic demand and supply is then covered by imports). Such a maximum could be imagined to be the consequence of a shortage of land, raw materials or (skilled) labour, the introduction of government or EEC measures to limit (over)production or reduce enviromental pollution or of the maximum capacities of certain plants which are usually changed with steps of complete units. We should therefore make two estimates: one of total production in a limited number of industries [4] and one of the final demand for the other industries (we shall not go into the problems created by making final demand dependent on price elasticity here; this problem often leads to iterative processes).

The second step entails the estimation of technological change. This means that the volume trends in intermediary consumption and primary costs are applied as factors to the intermediary and primary input coefficients in constant prices respectively. It should be mentioned in this respect that price changes (mainly of raw materials) dominated by market factors instead of by changes in production costs can be approached in the same way as technological changes in primary costs from a calculation-technical point of view.

With these two estimates, we have done our 'best' to define the whole system. The only thing left is to determine the missing figures. To this end we can make use of the selection vector technique[5].

We define the selection vector σ as follows

$$\sigma(k) = \begin{cases} 1 & k \in U \\ 0 & k \notin U \end{cases} \qquad (25)$$

and the residual vector ρ by

$$\rho = i - \sigma \qquad (26)$$

The set U then contains the ordinals of the industries for which total production is fixed (indeed, for convenience's sake we call U the set of the industries concerned).

The total production t_0^σ in prices of the base year 0 by industries indicated by set U is part of the total production t_0.

$$\begin{aligned} t_0^\sigma &= \hat{\sigma}\, t_0 \\ &= t_0 - \hat{\rho}\, t_0 \\ &= t_0 - t_0^\rho \end{aligned} \qquad (27)$$

Obviously

$$t_0^\sigma(k) = \begin{cases} t_0(k) & k \in U \\ 0 & k \notin U \end{cases} \qquad (28)$$

also applies.

The direct intermediary consumption, from the industries which do not belong to U, which is related to t_0^{σ} is given by

$$\hat{\rho}\ \tilde{A}_0\ t_0^{\sigma} \tag{29}$$

The total production t_0 leads to a consumption from the industries outside U. By way of the Leontief inverse we can calculate this cumulative consumption. However, in doing this we must exclude the commodities flows via U. Although a unit of output of the industries in the set U has a cumulative consumption itself, this consumption is only a <u>component</u> of the same total production precisely for which we are trying to calculate the cumulative consumption. The same goes for the cumulative consumption from the other industries behind the cumulative consumption of products from industries in U. To avoid double counting of this consumption, we must exclude these flows. The cumulative intermediary consumption $t_0^{\sigma}\ (\ t_0^{\sigma}\ )$ from industries outside set U is indicated to be a function of the total production t_0^{σ} by which it is induced, and is therefore given by

$$t_0^{\rho}\ (\ t_0^{\sigma}\ ) = (\ I - \hat{\rho}\ \tilde{A}_0\ \hat{\rho}\ )^{-1}\ \hat{\rho}\ \tilde{A}_0\ t_0^{\sigma} \tag{30}$$

This is of course also that part of total production t_0^{ρ} that is induced by the total production t_0^{σ}.

In addition to the term found above, there is another contribution to the total production t_0^{ρ}, viz. the production $t_0^{\rho}\ (\ y_0^{\rho}\ )$ which is connected with final demand for products from the industries outside U. So

$$t_0^{\rho} = t_0^{\rho}\ (\ y_0^{\rho}\ ) + t_0^{\rho}\ (\ t_0^{\sigma}\ ) \tag{31}$$

For the first term

$$t_0^{\rho}\ (\ y_0^{\rho}\ ) = (\ I - \hat{\rho}\ \tilde{A}_0\ \hat{\rho}\ )^{-1}\ y_0^{\rho} \tag{32}$$

applies. If we substitute (30) and (32) in (31), we get

$$t_0^\rho = (I - \hat{\rho}\, \tilde{A}_0\, \hat{\rho})^{-1} \{ \hat{\rho}\, \tilde{A}_0\, t_0^\sigma + y_0^\rho \} \tag{33}$$

for the total production by industries outside U. From (27) we then find the following for the total production t_0

$$t_0 = t_0^\sigma + (I - \hat{\rho}\, \tilde{A}_0\, \hat{\rho})^{-1} \{ \hat{\rho}\, \tilde{A}_0\, t_0^\sigma + y_0^\rho \} \tag{34}$$

If we now multiply the input coefficient by total production, we get the intermediary part of the IO-table we are looking for in prices of the base year

$$A_0 = \tilde{A}_0\, \hat{t}_0 \tag{35}$$

The corresponding vector of final expenditure follows from

$$y_0 = t_0 - A_0\, i \tag{36}$$

In view of its dual character, the new IO-table we have just found in constant prices can also be seen as a table in physical amounts, whereby the system of units is completely determined by the prices in the base year. By applying a gauge transformation with reference to the primary costs (this is equivalent with the Leontief price model), we get a table with shadow prices. Assuming the two applied estimates are correct, the shadow prices represent the price relations in the new year. The best estimate of the production structure leads to the so-called 'tallying' table, i.e. a table with identical row and column totals (cf. (8a) and (8b)), so that the smoothing RAS method with its 'polluting effect' is no longer necessary.

The importance of this method lies in the fact that a 'tallying' table is only needed in studies where price effects play an important role. The RAS method does not yield very correct price relations. The method with shadow prices yields the best possible estimates relative prices.

8. The IO-table as a set of relations

In the IO-table the consumption and production by industries is registrated; the destination of the production is also registrated. We can find all these data in matrix C (cf. (22)). Because evidently consumption by industries takes place to make production possible, input-output analysis starts out from a fixed relation between these quantities. In this sense, the IO-table represents a set of relations. In standard static input-output analysis, moreover, it is assumed that this table shows linear relations. This means that it is assumed that if production volume changes, consumption changes proportionally. This relation is reflected in the intermediary input coefficients $\tilde{A}$ and primary input coefficients $\tilde{P}$ respectively.

In practice, these relations are seldom proportional and often not even linear. As we know, expansion (scale enlargement) or reduction changes the production structure (i.e. the intermediary, primary and facet input coefficients) in different ways. Without going further into the complication involved here, this could be included under the denominator technological change.

Here we confine ourselves to the prortional relations. The input coefficients which proportionally relate volume to production and consumption respectively by way of the Leontief inverse, can also be used to relate other aspects of the production process, such as environmental pollution or price effects.

Let us trace the supposed effect of price changes. For row totals t_0 in the IO-table (matrix C) for base year 0, the following applies

$$t_0' = j' \, (I - \tilde{C})^{-1} \hat{t}_0 \tag{37}$$

Here, the vector j is a summation vector that works only on the primary costs: if we split j the same way as in C in (22), the first and last part consist only of zeros and the middle part only of ones. Equation (37) is in fact only another way of writing equation (10a), and is also the same as equation (15) if the vector s is based on a gauge on the basis of the sum of the primary costs.

Suppose now that precisely at time τ there is a simultaneous change in the production structure: a change of ΔA_0 in intermediary consumption and/or a change ΔP_0 in the primary costs, for example; these two can be combined in ΔC_0. Now

$$C_0^* = C_0 + \Delta C_0 \tag{38}$$

applies. The asterisk refers to the fact that C_0^* contains new amounts, but is still expressed in prices originally belonging to time 0.

The production totals will change due to two causes: first of all due to the changed intermediary demand and secondly due to the changed production structure which leads to new prices. If we express the changed production totals in new prices, we find

$$t'_\tau = j' \, (I - \tilde{C}_0^*)^{-1} \, \hat{t}_0 \tag{39}$$

Closer inspection of (39) shows that the vector t_τ is the gauge vector in aid of a gauge transformation on the basis of the formation of new value added. The vector with price ratios r' of the new prices with respect to the old ones follows directly

$$r' = j' \, (I - \tilde{C}_0^*)^{-1} \tag{40}$$

(This expression looks very similar to (14), the distinction between the expressions is due to theold input structure being expressed in new prices in (14) while a new structure is expressed in old prices in (40).

One often mentioned cause of inflation is that price increases lead to a form of compensation in wages. This wage compensation can in turn lead to price rises. This process is often referred to as the price-wage spiral. The wage compensation usually takes place on the basis of the so-called price index of consumption by households. This price index may be seen as an 'input structure' for wages. In this way, the above formulated form of the price-wage spiral can be simply simulated by including the relative composition of the relevant price index in the column corresponding with the row wages in matrix $\tilde{C}_0^*$: we then get matrix $\tilde{C}_0^1$. The effects of the price-wage spiral are then found by

$$r'_{sp} = j' \, (I - \tilde{C}_0^1)^{-1} \tag{41}$$

The form of expression (41) is the same as (24). Matrix $\tilde{C}_0^1$ contains a set of relations which describe the production process; matrix $\tilde{C}_0^1$ differs from $\tilde{C}_0^1$ in two respects. First of all the column totals of the input coefficients do not equal 1; remember, we are changing the production structure. In the second place, the elements in the column wages, i.e. the components of the price index, do not all equal zero. In fact we can consider matrix $\tilde{C}_0$ with the input coefficients as a special form of a more general matrix $\tilde{C}_0^1$ with relations on the production process.

Finally, there is the question of whether constructions such as the relation matrix $\tilde{C}_0^1$ actually have a Leontief inverse. In order to find that out, we have to go back to the composition of the partitioned matrix C in (22). For matrix $\tilde{C}_0^1$ we can construct the same form by transferring the row wages of the primary costs to the intermediary rows:

$$C_0^1 = \begin{bmatrix} A_0^1 & 0 & Y_A \\ P_0^1 & 0 & Y_P \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

In analogy with (23) the existence of the Leontief inverse of $\tilde{A}_0^1$ is sufficient for the existence of the Leontief inverse of $\tilde{C}_0^1$.

To see whether the Leontief inverse of $\tilde{A}_0^1$ exists, we partition $\tilde{A}_0^1$ into four parts

$$\tilde{A}_0^1 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \tag{43}$$

where $\tilde{A}_{11}$ contains the usual intermediary input coefficient, $\tilde{A}_{12}$ a column vector with the composition of the price index, $\tilde{A}_{21}$ a row vector with wage input coefficients and $\tilde{A}_{22}$ is the scalar zero. The Leontief inverse would then have to have the following form

$$(I - \tilde{A}_0^1)^{-1} = \begin{bmatrix} \tilde{L} + \tilde{L}\,\tilde{A}_{12}\,\xi^{-1}\,\tilde{A}_{21}\,\tilde{L} & \tilde{L}\,\tilde{A}_{12}\,\tilde{\xi}^{-1} \\ \xi^{-1}\,\tilde{A}_{21}\,\tilde{L} & \xi^{-1} \end{bmatrix} \tag{44}$$

where

$$\tilde{L} = (I - \tilde{A}_{11})^{-1} \tag{45}$$

and

$$\xi = (I - \tilde{A}_{22}) - \tilde{A}_{21}\,\tilde{L}\,\tilde{A}_{12} \tag{46}$$

The term $\tilde{L}$ is the usual Leontief inverse on the basis of input coefficients, which we know exists (i.e. is not singular). If we can prove that ξ^{-1} exists, then we also know that the Leontief inverse of $\tilde{C}_0$ exists.

In our case, $\tilde{A}_{22}$ equals 0. So

$$\xi = I - \tilde{A}_{21}\,\tilde{L}\,\tilde{A}_{12} \tag{47}$$

In a normal IO-table, the wage costs constitute part of the primary costs. From (10a) we know that in the vector with the cumulative primary costs all elements equal one. So for vector $\tilde{b}$, defined as

$$\tilde{b}' = \tilde{A}_{21} \tilde{L} \tag{48}$$

with the cumulative wage costs, all elements are smaller than one:

$$\tilde{b}_i < 1 \tag{49}$$

The sum of the elements of $\tilde{a}_i$ and the vecor A_{12} equals one

$$\sum_i \tilde{a}_i = 1 \tag{50}$$

so that

$$0 < \tilde{a}_i < 1 \tag{51}$$

also applies, since the price index is composed of more than one component unequal to zero. For the inproduct $\tilde{b}' \tilde{a}$, the following applies:

$$\begin{aligned} \tilde{b}' \tilde{a} &= \sum_i \tilde{b}_i \tilde{a}_i \\ &\leq \sum_i \max (\tilde{b}_i) \tilde{a}_i \\ &\leq \max (\tilde{b}_i) \sum_i \tilde{a}_i \\ &\leq \max (\tilde{b}_i) \\ &< 1 \end{aligned} \tag{52}$$

As I is a one-by-one matrix for the bottom right submatrix, and thus a scalar equal to one,

$$\xi > 0 \tag{53}$$

applies. Thus we have proven that ξ is nonsingular so that the Leontief inverse of $\tilde{C}_0^1$ does exist. The above proof for the case in which an extra relation is added to the primary costs can easily be extended to several relations. Obviously, it may sometimes be relevant to include other types of relationships in $\tilde{C}_0^1$; but it should be checked whether the expression in (46) is not singular.

A situation in which the wage-price spiral does not converge is a wage compensation on the basis of the total final demand while all primary cost categories increase in proportion with wage costs. In this situation, instead of (49), the following applies

$$b_i = 1 \tag{54}$$

The primary cost categories outside the wage costs thus have a damping effect (for that matter, domestic price rises will also have effects on exchange rates which in turn will affect import prices). Furthermore, in practice there is a damping effect of technological change which usually leads to greater efficiency (e.g. less labour per unit of product). The convergence of the wage-price spiral is also affected by deviations in the composition of the price index and the relative composition of the total final demand. This latter effect can have both an damping and an amplifying effect.

9. Facet requirements and cumulative facet costs

One popular application of input-output analysis is the calculation of facet requirements. Here we shall take as example energy requirements, i.e. cumulative energy consumption and the corresponding cumulative energy costs; we confine ourselves to electricity. We have opted for energy as the input-output analysis comprises some complications due to the fact that one energy carrier can be converted into another. For cumulative energy costs in particular it is not possible to use the input-output analysis in its most simple form. Among other things, these complications are related to the fact that when one energy carrier is converted into another, the energy content still available is reduced because of conversion losses, while the economic value of the new form of energy is higher. This higher value is due to the fact that the value of a product in itself is a cumulation of value added (eq. (10a)). Every production stage entails more value added which explains the above. The reduction of the energy content is inherent to physical or chemical processes.

In input-output analysis according to an open Leontief model it is usual to calculate the cumulative effects relating to a facet according to equation (5). For a facet such as energy consumption, where more than one energy carrier is involved, the row vector f' is replaced by a matrix E with energy consumption figures. In chapter 2 we already stated that a matrix with facet data shows a strong resemblance to the matrix of the primary costs. This suggests the construction of a new complet IO-table C_E including the facet energy consumption in analogy with (22). To this end we simply substitute matrix E for matrix P of the primary costs in (22), and similarly Y_E for Y_P. (Should we want to involve the primary costs in the cumulation at the same time, we can just combine them with E to form a larger matric; this also applies for any other facets. This extension is not relevant for our present line of argument.) The energy IO-table now obtained has the form

$$C_E = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 & Y_A \\ E & 0 & Y_E \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

In analogy with (23) we can calculate the Leontief inverse $\tilde{H}$ of the complete energy IO-table C_E

$$H = (I - \tilde{C}_E)^{-1}$$

$$= \begin{bmatrix} I - \tilde{A} & 0 & - \tilde{Y}_A \\ \tilde{P} & 0 & - \tilde{Y}_P \\ 0 & 0 & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I - \tilde{A})^{-1} & 0 & (I - \tilde{A})^{-1} \tilde{Y}_A \\ \tilde{E} (I - \tilde{A})^{-1} & I & \tilde{E} (I - \tilde{A})^{-1} \tilde{Y}_A + \tilde{Y}_E \\ 0 & 0 & I \end{bmatrix} \tag{56}$$

The term $\tilde{H}_{11}$ is again the usual Leontief inverse and $\tilde{H}_{21}$ contains the cumulative facet coefficients. As we already stated in chapter 6, there is essentially no difference between the terms $\tilde{H}_{11}$ and $\tilde{H}_{21}$; this is made clear in the form of the equations if we write $\tilde{H}_{11}$ as

$$\tilde{H}_{11} = I (I - \tilde{A})^{-1} \tag{57}$$

This form of $\tilde{H}_{11}$ is equivalent to the form of $\tilde{H}_{21}$ in (56), which implies a comparable role for A and E in (55).

A difference in comparibility between A and E can be seen by the fact that the second column of submatrices in (55) in principle zero. If we consider the facet energy consumption, it appears that this column is suddenly no longer zero in principle. All sorts of conversion processes turn out to occur, processes which can be described to function as an input structure for the consumed form of energy. The energy facet electricity consumption, for example, is closely related to the conversion of other energy carriers such as coal, natural gas, petroleum products etc. And this in turn can go even further back to the coversion of crude petroleum into petroleum products. Such conversion processes can be represented by choosing submatrix C_{22} suitably nonzero. However in converting energy carriers, nonenergy inputs are also consumed. This consumption can be found in A; later we shall make a reasonable case for this part of the input structure being assigned directly to the columns corresponding with the facets in E. The consequence of this is that the submatrix C_{12} also becomes nonzero. In consequence of the second column becoming nonzero, the complete Leontief inverse has the following form

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} \\ 0 & 0 & I \end{bmatrix} \tag{58}$$

where all submatrices of the first two rows are nonzero.

In this Leontief inverse we find both the cumulative consumption from the original and the pseudo industries (of the facets). If we want find out what the cumulative energy consumption is, we first of all want to take the sum of the cumulative consumption of the various energy carriers. This will lead to great errors, however, as we would then take both the cumulative consumption of both so-called primary energy carriers and the forms obtained by conversion together. Even if we only add together the cumulative consumption of primary energy carriers things could go wrong, since an energy form such as electricity, which usually is considered to be nonprimary, is sometimes obtained without there being a (conventional) primary energy carrier as a basis; this directly produced electric energy may then be 'forgotten'. This is the case for electricity generated by nuclear energy, water power, wind energy, solar energy, waste incineration etc.; this part of electricity is in fact a primary energy carrier. Moreover, it does not provide desired information with respect to conversion products.

Let us now consider the production structure with respect to the electricity production and consumption and see how these and other pitfalls can be avoided in energy analysis. Normally in an IO-table we would find a row of electricity producers in which the values of the products consumed by the various buyers are mentioned. In the corresponding column we find the consumption by electricity producers. Further, we assume that in a row of E (and thus also Y_E) we find the energy consumption in energy terms.

For an accurate energy analysis it is necessary to 'homogenize' these rows of pseudo industries. Electricity producers often provide nonenergy products (sometimes services); for example: sulphuric acid, gypsum, tar, cinders, fly ash, computer services, installation services etc. In as far as these nonenergy products have to be considered as unavoidably connected with the production of electricity, they should be included on one or more separate rows. A number of the inputs have to be transferred to the corresponding columns. The remaining energy row must be further homogenized by distinguishing between various energy products: electricity, steam and warm water; the input structure will also have to be divided proportionally. At the same time, the row will have to be divided according to these energy carriers.

In connection with the enormous price differences charged to the various buyers, the electricity row must be split into at least two rows: one for distribution services and one for the actual electricity supply. The column belonging to the former of these contains the input structure for - among other things - administrative costs and the maintenance and depreciation of the intricate distributon network. The second column contains the depreciation of the generators and the high-tension network and of course the costs of fuel and consumption of fuel in energy terms. Even so, price differences can occur due to all sorts of causes which leads to buyers being charged for different amounts of energy for the same amount of electricity (though this may sometimes be justified by the difference between continual and peak buyers). This is usually unjustified and to avoid it, it would be better to transfer the input structure from the column corresponding with the row of electricity in values to the column corresponding with electricity in amounts of energy.

The electricity rows (values and amounts) are then each split into at least two rows: electricity from conversion and generated (primary) electricity. The latter group includes nuclear energy, water energy, wind energy, solar energy and waste incineration. If desired, these groups can be further divided according to the various fuels or techniques used. The rows can be obtained by means of proportional division from the original sum rows; if information that certain customers are mainly supplied with electricity produced in a specific way is available, this can also be included in the rows if desired. The columns are obtained from information about the input structures for the various techniques.

The transfer of the input structures to the columns corresponding with the rows in amounts means that submatrix C_{12} no longer consists only of zeros. For the other energy products, too, it would be better to transfer input structures to the energy columns. In brief, this all leads to the following breakdowns with respect to electricity production:

Table 1. Breakdown of electricity costs and consumption

costs/consumption of	matrix	unit	column
1. nonenergy products	A	money	filled
2. steam	A	money	empty
3. warm water	A	money	empty
4. distribution costs	A	money	filled
5. electricity costs	A	money	empty
6. steam	E	energy	filled
7. warm water	E	energy	filled
8. electricity from conversion	E	energy	filled
9. elecrticity from generation	E	energy	filled

The same specifications should be applied to the rows and columns of other energy conversion industries in A: petroleum refineries, coke manufacturers, coal-gas manufacturers etc. A further specific problem in the energy sector concerns industries which, although not by nature energy manufacturers, produce energy products as a consequence of their specific production processes. This situation often occurs at blast furnaces, the petro-chemical industry, artficial fertilizer manufacturers etc. It is very difficult to assign an input structure to these energy products, moreover it does not make any sense to assign large scale iron-ore consumption to the product of electricity (except the ore required for capital goods). A normal manufacturer would produce these energy products in a completely different way. These products should indeed be seen as a saving , i.e. negative consumption of the products. Incidental buyers of the products can be assigned a normal cumulative input structure. In the case of blast furnaces, there is another specific matter: here blast furnace mixed gas is produced, although undoubtedly not deliberately. Here it should be seen as a negative consumption of the product which it substitutes; usually natural gas or coal gas but sometimes coal or fuel oil.

Imported energy constitutes a special problem. Sometimes it can be made to compete with domestic production and can then be dealt with in the normal manner in the framework of the IO-table. More often, however, it is not possible to have these energy carriers compete with domestic products because the energy carrier concerned is not produced domestically or for other reasons: domestic production may not in principle be able to meet demand, for example, while the domestic input structure deviates greatly from the foreign one. This is the case for North Sea oil versus crude petroleum from the Middle East among other things. In such cases, two sets of two rows each should be added to the table together with the corresponding columns. The first set contains a row for submatrix A with imported (cif) values and one for submatrix E in energy terms. The first coulmn remains empty, the other is filled with transport costs, the values at producer prices and energy amounts. The last two items are entered on the rows of the second set. The second set also contains a row for in A and one for in E; except for the two formerly mentioned items, they remain empty. The column in the second set corresponding with the values is filled with the (foreign) input structure and the other in the second set remains empty.

In this way we have homogenized IO-table C with respect to energy consumption, and thus obtained IO-table C_E. With this IO-table we can calculate the Leontief inverse of (58). This Leontief inverse now gives the cumulative consumption coefficient of the products of both the original and the pseudo industries. If we want to find out the cumulative energy consumption, we simply sum up the primary energy carriers (including generated electricity).

For some purposes, it is necessary to know the cumulative actual energy consumption: the consumption of energy <u>not</u> used for direct conversion. In that case, first of all, <u>all</u> energy carriers are summed up, after which the part used for direct conversion is subtracted. The calculation of what is to be subtracted seems complicated, for while we do want to subtract the energy content of, say, converted petroleum products, we may not want to subtract the cumulative energy consumption involved in transport to and from refeineries etc. To this end we regard submatrix C_{22} that describes the conversion processes. We construct matrix Ω with the same dimensions as C_{22}, which is completely filled with zeros except for the part which does not refer to energy conversion (e.g. energy supplies for transport to petroleum refineries). The pure conversion matrix Z then follows from

$$Z = C_{22} - \Omega \tag{59}$$

If we define the corresponding totals t_2 as

$$t_2 = C_{21}\, i_1 + C_{22}\, i_2 + C_{23}\, i_3 \tag{60}$$

where i_1, i_2 and i_3 respectively are summation vectors with the correct length. We then calculate the coefficient matrix by

$$\tilde{Z} = Z\, \hat{t}_2^{\,-1} \tag{61}$$

Depending on what we do and what we do not define as belonging to Z, complicated structures may occur in the pure conversion matrix. The energy input which must be assigned to the conversion processes can be found in a new Leontief inverse $\tilde{L}_Z$

$$\tilde{L}_Z = (I - \tilde{Z})^{-1} \tag{62}$$

The cumulative actual energy consumption per unit of product is now

$$\tilde{W} = (\tilde{H}_{21}, \tilde{H}_{22} - \tilde{L}_Z, \tilde{H}_{23}) \tag{63}$$

Summation over the various energy carriers leads to

$$\tilde{w} = i' \tilde{W} \tag{64}$$

The latter summation can obviously only be introduced if energy consumption is recorded in the same unit on the various rows in C_E.

We then consider the cumulative energy costs. Of the 9 rows in table 1, rows 2 - 5 are usually regarded as energy costs. However, some people will not see row 4 as energy costs. Such considerations apply to all energy carriers. With respect to the composition of energy costs, it can be said that a significant part of energy costs related to higher energy forms, such as electricity, concern the consumption of other energy carriers. In addition there are nonenergy costs, but again these hide cumulative energy costs. If we calculate the cumulative energy costs by way of the Leontief inverse for C_E i.e. from $\tilde{H}$, a number of energy costs are included more than once, which is obviously not correct. We can simply state that in order to arrive at a correct cumulative costs analysis, no energy costs composed of cumulative energy costs are to be included in the calculation! This implies that these energy costs - situated further back - of the input structure concerned have to be excluded. The input structures of the rows 2 - 5 in table 1 are stated in columns 4 and 6-9. We must render these columns zero; for this we have the selection vector mechanism at our disposal [5].

We define selection vector σ_E by setting the elements relating to the energy input structure at one and the other elements at zero. For the residual vector ρ_E we then find

$$\rho_E = i - \sigma_E \tag{65}$$

We then render the desired columns of C_E zero according to the expression

$$\hat{\rho}_E \, C_E \tag{66}$$

and find the cumulative energy costs in the expression

$$(I - \hat{\rho}_E \, \tilde{C}_E)^{-1} \tag{67}$$

Sometimes we may be interested in finding out the cumulative primary energy costs against producer prices. These concern the cumulative products of the energy generating plants. These kinds of plants, too, have a cumulative energy consumption with costs that can be assigned to it. The latter cost structure is already incorporated in the price of the primary energy carriers, however. In analogy with the foregoing, this implies that this input structure must be eliminated with the aid of selection vectors. We select the energy generation (pseudo) industries with the aid of selection vector σ_D; this leads to the complementary residual vector ρ_D. The cumulative primary energy costs then follow from

$$(I - \hat{\rho}_D \, \tilde{C}_E)^{-1} \tag{68}$$

We now get the curious phenomenon that the cumulative energy costs refer to cumulative actual energy consumption, but also to the cumulative primary energy consumption (i.e. the costs of making energy available). Moreover, the cumulative primary energy costs also refer to the cumulative primary energy consumption, while there is also the option of considering the cumulative consumption of primary energy behind that as well! For there are also cumulative energy costs at the basis of cumulative energy costs!

The model described here in aid of energy consumption and energy costs can also be applied to other facets. Consumption and costs of raw materials for example, though there is the extra complication that the various raw materials cannot usually be added together.

The conclusions of this chapter are:

1. For energy analysis the IO-table should be extended with the energy consumption in energy terms.
2. The complete IO-table must be homogenized.
3. The input structure for the conversion plants should be divided between the columns corresponding with the rows of the energy costs and those of energy consumption; they will usually have to be transferred completely to the last columns as the production relations are then reflected better.
4. The energy link products should be regarded as negative consumption.
5. To calculate total consumption the rows of primary energy consumption in the complete Leontief inverse should be added together.
6. For the cumulative actual energy consumption the result of a limited number of cumulation processes should be subtracted.
7. The cumulative (actual or otherwise) energy costs can only be calculated from a Leontief inverse when an appropriate selection vector is applied.

10. The IO-table as a time operator

The Leontief inverse can be turned into a Taylor series expansion quite simply

$$(I - \tilde{A})^{-1} = \sum_{i=0}^{\infty} \tilde{A}^{i}$$

$$= I + \tilde{A} + \tilde{A}^{2} + \tilde{A}^{3} + \ldots \qquad (69)$$

The interpretation of this series expansion is that the zeroth order term represents the production to which one wants to link a certain effect, and is therefore normalized to one unit. The first term represents what has happened one step back in the production process (this is therefore the direct consumption); the second term represents what has happened two steps back in the production process, etc. This chain of production steps suggests that the terms of the series expansion are related to the passing of time, a suggestion made in the study on production chains [6], for example, but also to be found in work by earlier researchers. It is obvious that the higher order terms reflect processes which precede certain processes reflected in lower order terms. However, instinct tells us that the processes represented by a higher order term did not all take place in the same time interval; for as different processes require different amounts of time, time shifts will occur in higher order terms.

For a better insight we shall examine the individual production steps in relation to the time they take. In one production step we encouter roughly the following phases: 1. supply of raw materials; 2. storage of raw materials (buffer stock); 3. actual processing; 4. stock formation of products, followed by actual delivery. Let us now define the time needed for one production step as the time that passes between the supply by the previous industry and the delivery by the industry under review. In this way we consistently regard all production phases in relation to the time they take for each production process.

The time the first phase - the supply of raw materials - takes varies considerably depending on the nature of the raw materials concerned. Input such as electricity involves no transport time at all, as the producer supplies it on the customer's premises; coconuts from a Pacific island, on the other hand, will undergo a few month's journey before they arrive at the customer's (for some time effect studies it would probably be better to define the time of supply of imported commodities as the time at which they pass through customs). The time needed for transport appears to depend mainly on the nature of the supplier, but sometimes the nature of the buyer is also important. Some commodities are sold via wholesale, where stock formation is a common occurrence, which then takes extra time. If large buyers do not buy through wholesalers, while small buyers do, the nature of the buyer is relevant for the time involved. The second phase - storage - turns out to depend on both the nature of the commodities (i.e. the nature of supplier) and the nature of the buyer. By definition, a commodity like electricity is not stored. If storage does take place then a minimum storage time must apply, as many buyers keep a buffer stock; afterwards commodities that arrive at the same time are consumed over a certain time interval. Large buyers usually recieve their raw materials more frequently than small buyers and this obviously also influences the time required. Many buyers are large buyers of a number of raw materials and small buyers of others. Here both the nature of the commodities and the nature of the buyer affect the time involved. The amount of time involved in the third phase, the actual production step, depends on the type of process. Some processes are fairly direct, others take several months, agricultural production for example. The length of time involved in the fourth phase - stock formation and delivery - also varies widely. Electricity, for example, cannot be stored, while a uranium enriching plant supplies its products at very great intervals.

We see then that the time that passes between the supply of a raw material and the delivery of a product in which it is processed is not determined onesidedly by the supplying or buying industry, but by a combination of supplier and buyer. Moreover, the time intervals are also influenced by the nature of the phenomenon under study: the time intervals differ for the actual flow of commodities through the production structure, the incorporation of price changes of raw materials (some price changes are incorporated immediately while others are only once the stock at the old price has been processed, still others are saved up and incorporated collectively after a certain time interval), or a period of strong disruption when the supply of essential raw materials ceases temporarily. Following from chapter 8, we shall take the example of incorporating prices changes here.

Let us go back to the series expansion in (69). We want to consider every term as an <u>equal</u> time interval for <u>all</u> rows in A. Ideally we would choose the interval infinitesimally small, but this is impossible in practice. Moreover, the information about the time concerned is usually so inaccurate that such precision would not be meaningful. Every term in the series expansion must describe what happens in the successive periods. Suppose (initial) price changes occur in what we call the zeroth period. Some of these price changes are incorporated quickly i.e. between 0.5 and 1.5

periods later. (We assume here that no price changes are incorporated in the interval 0 - 0.5, as the actual production time usually leads to a certain 'reaction time'. In principle, the model can be adapted to extremely quick price change incorporations.) We shall assume that for these price mutations there is a reaction time of one period on average. However, no other price changes are incorprated in this time interval.

In the interval of 1.5 - 2.5 periods later, a number of price changes from the first period are incorporated, but also some of the price changes of the zeroth period that had not yet been incorporated. And this chain of price adjustments extends across several time intervals; formally speaking, across an infinite number of intervals.

All the price changes from the previous interval are incorporated in the series expansion of (69), there is one 'standard' reaction time. This standard reaction time does not correspond with the practice in which the reaction time for every cell of A can differ. However, we can construct a matrix A_δ in which we have all the relations that fit in the 'standard' time intervals of one period. This requires the creation of so-called 'waiting rooms': first, second, third, etc. waiting rooms in order to be able to reflect the various delays. Each industry requires a different number of waiting rooms, depending on the maximum number of standard intervals delay that can occur. To this end, for each industry, we introduce an extra column for each waiting room, with a corresponding row in the table. The original consumption by a certain industry must now be spread out across the set of waiting rooms which now represent an industry. Every waiting room further has an output that equals the sum of the inputs supplied completely to the waiting room of a lower order. Only the waiting room with the lowest order has an output to the waiting rooms of other industries or to the final demand.

The thus obtained matrix A_δ with elements $\alpha_{i_\lambda j_\mu}$ has much larger proportions than the original matrix A. Subindices λ and μ give the order of the waiting rooms and thus the delays. The waiting rooms function as a sort of pseudo industry here.

We can thus consider the original IO-table as a projection of the table with waiting rooms. Between elements a_{ij} of the intermediary part A of the original table and the elements $\alpha_{i_\lambda j_\mu}$ of the intermediary part A_δ of the new table there are the following relations:

$$\alpha_{i_1 j_\mu} = \varphi_{ij\mu}\, a_{ij} \tag{70}$$

$$\sum_\mu \varphi_{ij\mu} = 1 \qquad \Rightarrow \qquad \sum_\mu \alpha_{i_1 j_\mu} = a_{ij} \tag{71}$$

$$\begin{aligned} \alpha_{j_\lambda j_\mu} &= \sum_i a_{ij} && \lambda - \mu = 1 \quad \lambda = \lambda_{max}(\ j\ ) \\ &= \sum_i \alpha_{i_1 j_\lambda} + \alpha_{j_{\lambda-1} j_\lambda} && \lambda - \mu = 1, \quad \lambda < \lambda_{max}(\ j\ ) \\ &= 0 && \lambda - \mu \neq 1 \end{aligned} \tag{72}$$

$$\alpha_{i_\lambda j_\mu} = 0 \qquad \lambda > 1, \quad i \neq j \tag{73}$$

Equation (70) states that an element of the original matrix A is spread out across several waiting rooms, while (71) states that the amount (value) spread out must equal the original one. Equation (72) states that the whole consumption package is supplied from a waiting room of a higher order to one with an order exactly one lower; $\lambda_{max}(\ j\ )$ gives the total number of

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Figure 2. The composition of matrix A_δ (top) in relation to the original matrix (bottom). Part of the column and row respectively of industries i and j are shown. Industry i is divided into three waiting rooms, industry j into four. The original cells of matrix A are turned into (3x3), (3x4), (4x3) and (4x4) submatrices of A_δ respectively.

	$a_{ii} = \sum_{\mu} \alpha_{i_1 i_\mu}$	#	$a_{ij} = \sum_{\mu} \alpha_{i_1 j_\mu}$	
	#	#	#	
	$a_{ji} = \sum_{\mu} \alpha_{j_1 i_\mu}$	#	$a_{jj} = \sum_{\mu} \alpha_{j_1 j_\mu}$	

		$\alpha_{i_3 i_2}$						
			$\alpha_{i_2 i_1}$	#				
	$\alpha_{i_1 i_3}$	$\alpha_{i_1 i_2}$	$\alpha_{i_1 i_1}$		$\alpha_{i_1 j_4}$	$\alpha_{i_1 j_3}$	$\alpha_{i_1 j_2}$	$\alpha_{i_1 j_1}$
		#		#			#	
						$\alpha_{j_4 j_3}$		
							$\alpha_{j_3 j_2}$	
				#				$\alpha_{j_2 j_1}$
	$\alpha_{j_1 i_3}$	$\alpha_{j_1 i_2}$	$\alpha_{j_1 i_1}$		$\alpha_{j_1 j_4}$	$\alpha_{j_1 j_3}$	$\alpha_{j_1 j_2}$	$\alpha_{j_1 j_1}$

waiting rooms for industry j. Equation (73) states that no transfers are possible from waiting rooms of a higher order to waiting rooms belonging to another industry. Figure 2 shows how matrix A_δ is filled.

In analogy with usual input-output analysis, we can calculate the input coefficient $\tilde{A}_\delta$ from the matrix A_δ. For this we must divide each individual column by the corresponding totals t_δ, where the elements of t_δ for the ordinary industries equal the corresponding elements of t, and for the waiting rooms equal the row totals.

If we now apply the series expansion of (69) to matrix $\tilde{A}_\delta$ we can follow the commodities flow through the production structure on the basis of equal time intervals. Here matrix $\tilde{A}_\delta$ functions as a time operator, whereby a higher power of $\tilde{A}_\delta$ represents a repeated application of the time operator. If we want to measure the rate of inflation and perhaps damping times, we must extend matrix $\tilde{A}_\delta$ with an input structure for wages, in analogy with chapter 8, but this time extended with the waiting rooms which describe the average delay of the wage rises granted per industry. In this way we get matrix $\tilde{A}^1_\delta$.

In order to follow inflation through time, it is necessary to determine the terms in the series expansion individually. Although in the usual Leontief inverse the progression coverges relatively quickly, this will not be the case for terms ${\tilde{A}^1}^i$. This is due first of all to the addition of the input structure for wage costs and secondly to the addition of waiting rooms. This leads to the possibility of computer machine errors cumulating in an undesired way, in addition to a great deal of computer time. The cumulation of machine errors can be restriced somewhat by calculating the various terms of the series expansion by multiplication of previously obtained terms in such a way that only the terms with an exponent which can be written as a power of 2 are used. The computer time can be reduced somewhat by making use of techniques for sparse matrices for lower powers. Whether or not sufficient convergence is obtained with a certain term can be checked with the aid of the Leontief inverse

$$\tilde{L}^1_\delta = (I - \tilde{A}^1_\delta)^{-1} \tag{74}$$

which is the limit of the summation of the terms of the series expansion. When the convergence takes place very slowly, increasing the time interval for higher order terms by consistently leaving out one or more terms of the series expansion can be considered.

Of course, the question arises what the average delay for the incorporation of price changes is. The sum of all the delays measured in the time intervals connected with the waiting rooms is given by the series expansion

$$\begin{aligned} D &= 0 \;.\; I + 1 \;.\; \tilde{A}^1_\delta + 2 \;.\; {\tilde{A}^1_\delta}^2 + 3 \;.\; {\tilde{A}^1_\delta}^3 + \ldots \\ &= \sum_{i=0}^{\infty} i \;.\; {\tilde{A}^1_\delta}^i \\ &= (I - \tilde{A}^1_\delta)^{-1} \{ (I - \tilde{A}^1_\delta)^{-1} - I \} \\ &= \tilde{L}^1_\delta (\tilde{L}^1_\delta - I) \end{aligned} \tag{75}$$

The average delay $\bar{D}$ is now obtained by dividing each element by the cumulative contribution of one (pseudo) industry to the other:

$$\bar{D} = D \oslash \tilde{L}^1_\delta \tag{76}$$

With the aid of the projection of the terms of the series expansion on a time axis we can follow the consequences of changes in the production structure through time. For this we use the complete IO-table of (22) as a starting point, whereby primary inputs and the final output are considered as an intrinsic part of the production structure. We expand this matrix $\tilde{C}$ with waiting rooms to matrix $\tilde{C}_\delta$. In adding the waiting rooms it should be kept in mind that the delays here can be influenced, as some companies are able to anticipate (somewhat) changes in the production structure. The consequences of changes in the production structure probably spread out more quickly due to this than the actual flow rate of commodities through the same production structure. For that matter, it should be kept in mind that a change in the production structure usually induces further changes. For changes in consumption and production often lead to undesired stock changes which have to be disposed of.

A change in the production structure at time 0 with a term $\Delta\tilde{C}_\delta$ (a change in the flow rate between two industries is therefore also a change in the production structure) leads to the new production structure:

$$\tilde{C}_\delta + \Delta\tilde{C}_\delta \tag{77}$$

The cumulative effects of a production structure noticed after n intervals are:

$$\sum_{i=0}^{n} \tilde{C}_\delta^{\,i} \tag{78}$$

while for the new structure they are

$$\sum_{i=0}^{n} (\tilde{C}_\delta + \Delta\tilde{C}_\delta)^i \tag{79}$$

The change in effects then noticed is the addition of the following term of the series expansion according to the new structure minus the addition which should have taken place according to the old structure:

$$\Delta_n = (\tilde{C}_\delta + \Delta\tilde{C}_\delta)^{n+1} - \tilde{C}_\delta^{\,n+1} \tag{80}$$

Further elaboration gives

$$\Delta_n = \sum_{i=0}^{n+1} \begin{bmatrix} n+1 \\ i \end{bmatrix} \Delta\tilde{C}_\delta^{\,i}\, \tilde{C}_\delta^{\,n+1-i} - \tilde{C}_\delta^{\,n+1}$$

$$= \sum_{i=1}^{n+1} \begin{bmatrix} n+1 \\ i \end{bmatrix} \Delta\tilde{C}_\delta^{\,i}\, \tilde{C}_\delta^{\,n+1-i} \tag{81}$$

For most of the changes in the production structure it can be said that only the first term, or the first two terms at the most, of (81) are relevant. We can therefore state that a change in the production structure after n time intervals still leads to the following changes:

$$\Delta_n \approx (n + 1) \Delta\tilde{C}_\delta \tilde{C}_\delta{}^n + \frac{1}{2} n (n + 1) \Delta\tilde{C}_\delta{}^2 \tilde{C}_\delta{}^{n - 1} \tag{82}$$

The time operator $\tilde{A}_\delta^{-1}$ used here has no possiblity of simulating the extremely quick passing on of price change effects. The effects of such an extremely quick passing on can be represented by an operator Γ which is considered as <u>not</u> taking any time at all, and which thus works <u>immediately</u>. The processes that take time <u>cannot</u> therefore be simulated by this operator. This means that this operator may not have a resultant in its effects on the columns belonging to the processes that take time; in the corresponding columns of Γ we indeed find only zeros. The processes that do work immediately must be represented by this operator. For the immediate processes we find in the columns of Γ the part of the input structure that represents the immediate process. Because of the special form of Γ, this matrix is idempotent from a certain power k + 1 onwards, and indeed in such a way that the resulting matrix consists solely of zeros:

$$\Gamma^{i + 1} = \Gamma^i = \Gamma^{k + 1} = 0 \qquad i > k \tag{83}$$

while

$$\Gamma^i \neq 0 \qquad i \leq k \tag{84}$$

also applies. Because it is idempotent, this indicates that more than one immediate process can take place successively (i.e. with an infinitesimally small time interval). In realistic cases, k will be relatively small as it is very unprobable that many processes which are 'immediate' by approximation can take place within the standard time interval. The operator Γ can therefore only approximate reality if no production loops occur. A realistic matrix Γ will be all but completely filled with zeros.

As part of the input structure is now mentioned in Γ, this part must be eliminated from $\tilde{A}^1_\delta$. This often means that certain elements have to be rendered zero. We call this new time operator from which the immediate processes have been eliminated $\tilde{A}^{1*}_\delta$.

The series expansion of (75) which represents the time lag is now transformed to

$$\bar{D}^* = \sum_{i=0}^{\infty} i \, . \left[\tilde{A}^{1*}_\delta + \sum_{j=1}^{k} \Gamma^j \right]^i$$

$$= [I - (I - \Gamma)^{-1} \tilde{A}^{1*}_\delta]^{-1} [\{ I - (I - \Gamma)^{-1} \tilde{A}^{1*}_\delta \}^{-1} - I] \quad (85)$$

The interpretation of this series expansion is as follows: in every time interval a number of processes are carried out, represented by the time operator $\tilde{A}^{1*}_\delta$ followed by an infinite number of immediate processes of which only the first k have any influence, represented by operators Γ^j.

The influence of changes in the production structure which are of consequence for the capital goods must be calculated separately. The reason behind this is that time lags here are much greater than those involved in ordinary intermediary consumption; therefore it would be necessary to consider many more terms in the series expansion. The effects via the consumption of capital goods must be simulated by creating a separate time scale for them, whereby the other effects are considered as having a (nearly) immediate effect.

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Available National Accounts Occasional Papers

NA/01 *Flexibility in the system of National Accounts*, Eck, R. van, C.N. Gorter and H.K. van Tuinen (1983)
This paper sets out some of the main ideas of what gradually developed into the Dutch view on the fourth revision of the SNA. In particular it focuses on the validity and even desirability of the inclusion of a number of carefully chosen alternative definitions the "Blue Book", and the organization of a flexible system starting from a core that is easier to understand than the 1968 SNA.

NA/02 *The unobserved economy and the National Accounts in the Netherlands, a sensitivity analysis*, Broesterhuizen, G.A.A.M. (1983)
This paper studies the influence of fraud on macro-economic statistics, especially GDP. The term "fraud" is used as meaning unreporting or underreporting income (e.g. to the tax authorities). The conclusion of the analysis of growth figures is that a bias in the growth of GDP of more than 0.5% is very unlikely.

NA/03 *Secondary activities and the National Accounts: Aspects of the Dutch measurement practice and its effects on the unofficial economy*, Eck, R. van (1985)
In the process of estimating national product and other variables in the National Accounts a number of methods is used to obtain initial estimates for each economic activity. These methods are described and for each method various possibilities for distortion are considered.

NA/04 *Comparability of input-output tables in time*, Al, P.G. and G.A.A.M. Broesterhuizen (1985)
In this paper it is argued that the comparability in time of statistics, and input-output tables in particular, can be filled in in various ways. The way in which it is filled depends on the structure and object of the statistics concerned. In this respect it is important to differentiate between coordinated input-output tables, in which groups of units (industries) are divided into rows and columns, and analytical input-output tables, in which the rows and columms refer to homogeneous activities.

NA/05 *The use of chain indices for deflating the National Accounts*, Al, P.G., B.M. Balk, S. de Boer and G.P. den Bakker (1985)
This paper is devoted to the problem of deflating National Accounts and input-output tables. This problem is approached from the theoretical as well as from the practical side. Although the theoretical argument favors the use of chained Vartia-I indices, the current practice of compilating National Accounts restricts to using chained Paasche and Laspeyres indices. Various possible objections to the use of chained indices are discussed and rejected.

NA/06 *Revision of the system of National Accounts: the case for flexibility*, Bochove, C.A. van and H.K. van Tuinen (1985)
This paper examines the purposes of the SNA and concludes that they frequently conflict with one another. Consequently, the structure of the SNA should be made more flexible. This can be achieved by means of a system of a general purpose core supplemented with special modules. This core is a full-fledged, detailed system of National Accounts with a greater institutional content than the present SNA and a more elaborate description of the economy at the meso-level. The modules are more analytic and reflect special purposes and specific theoretical views. It is argued that future revisions will concentrate on the modules and that the core is more durable than systems like present SNA.

NA/07 *Integration of input-output tables and sector accounts; a possible solution*, Bos, C. v.d. (1985)
In this paper, the establishment-enterprise or company problem is tackled by taking the institutional sectors to which the establishments belong into account during the construction of input-output tables. The extra burden on the construction of input-output tables resulting from this approach is examined for the Dutch situation. An adapted sectoring of institutional units is proposed for the construction of input-output tables. The proposed approach contains perspectives on further specification of the institutional sectors,

households and non-financial enterprises and quasi-corporate enterprises.

NA/08 *A note on Dutch National Accounting data 1900-1984*, Bochove, C.A. van (1985)
This note provides a brief survey of Dutch national accounting data for 1900-1984, concentrating on national income. It indicates where these data can be found and what the major discontinuities are. The note concludes that estimates of the level of national income may contain inaccuracies; that its growth rate is measured accurately for the period since 1948; and that the real income growth rate series for 1900-1984 may contain a systematic bias.

NA/09 *The structure of the next SNA: review of the basic options*, Bochove, C.A. van and A.M. Bloem (1985)
There are two basic issues with respect to the structure of the next version the UN System of National Accounts. The first is its 'size ': reviewing this issue, it can be concluded that the next SNA must be 'large ' in the sense of containing an integrated meso-economic statistical system. It is essential that the next SNA contains an institutional system without the imputations and attributions that pollute present SNA. This can be achieved by distinguishing, in the central system of the next SNA, a core (the institutional system), a standard module for non-market production and a standard module describing attributed income and consumption of the household sector.

NA/10 *Dual sectoring in National Accounts*, Al, P.G. (1985)
The economic process consists of various sub-processes, each requiring its own characteristic classification when described from a statistical point of view. In doing this, the interfaces linking the sub-systems describing the individual processes must be charted in order to reflect the relations existing within the overall process. In this paper, this issue is examined with the special refernce to dual sectoring in systems of National Accounts. Following a conceptual explanation of dual sectoring, an outline is given of a statistical system with complete dual sectoring in which the linkages are also defined and worked out. It is shown that the SNA 1968 is incomplete and obscure with respect to the links between the two sub-processes.

NA/11 *Backward and forward linkages with an application to the Dutch agro-industrial complex*, Harthoorn, R. (1985)
Some industries induce production in other industries. An elegant method is developed for calculating forward and backward linkages avoiding double counting. For 1981 these methods have been applied to determine the influence of Dutch agriculture in the Dutch economy in terms of value added and labour force.

NA/12 *Production chains*, Harthoorn, R. (1986)
This paper introduces the notion of production cains as a measure of the hierarchy of industries in the production process. Production chains are sequences of transformation of products by successive industries. It is possible to calculate forward transformations as well as backward ones.

NA/13 *The simultaneous compilation of current price and deflated input-output tables*, Boer, S. de and G.A.A.M. Broesterhuizen (1986)
This paper discusses a number of aspects of the procedure according to which input-output tables are compiled in the Netherlands. A few years ago this method underwent an essential revision. The most significant improvement means that during the entire statistical process, from the processsing and analysis of the basic data up to and including the phase of balancing the tables, data in current prices and deflated data are obtained simultaneously and in consistency with each other. Data in current prices first used to be compiled and data in constant prices and changes in volume and prices used to be estimated only afterwards. With the new method the opportunity for the analysis of the interrelations between various kinds of data, and thus better estimates is used.

NA/14 *A proposal for the synoptic structure of the next SNA*, Al, P.G. and C.A. van Bochove (1986)

NA/15 *Features of the hidden economy in the Netherlands*, Eck, R. van and B. Kazemier (1986)
This paper presents survey results on the size and structure of the hidden labour market in the Netherlands.

NA/16 *Uncovering hidden income distributions: the Dutch approach*, Bochove, C.A. van (1987)

NA/17 *Main national accounting series 1900-1986*, Bochove, C.A. van and T.A. Huitker (1987)
The main national accounting series for the Netherlands, 1900-1986, are provided, along with a brief explanation.

NA/18 *The Dutch economy, 1921-1939 and 1969-1985. A comparison based on revised macro-economic data for the interwar period*, Bakker, G.P. den, T.A. Huitker and C.A. van Bochove (1987)

NA/19 *Constant wealth national income: accounting for war damage with an application to the Netherlands,1940-1945*,Bochove, C.A. van and W. van Sorge (1987)

NA/20 *The micro-meso-macro linkage for business in an SNA-compatible system of economic statistics*, Bochove, C.A. van (1987)

NA/21 *Micro-macro link for government*, Bloem, A.M. (1987)
This paper describes the way the link between the statistics on government finance and national accounts is provided for in the Dutch government finance statistics.

NA/22 *Some extensions of the static open Leontief model*, Harthoorn, R. (1987)
The results of input-output analysis are invariant for a transformation of the system of units. Such transformation can be used to derive the Leontief price model, for forecasting input-output tables and for the calculation of cumulative factor costs. Finally the series expansion of the Leontief inverse is used to describe how certain economic processes are spread out over time.

NA/23 *Compilation of household sector accounts in the Netherlands National Accounts*, Laan, P. van der (1987)
This paper provides a concise description of the way in which household sector accounts are compiled within the Netherlands National Accounts. Special attention is paid to differences with the recommendations in the United Nations System of National Accounts (SNA).

NA/24 *On the adjustment of tables with Lagrange multipliers*, Harthoorn, R. and J. van Dalen (1987)
An efficient variant of the Lagrange method is given, which uses no more computer time and central memory then the widely used RAS method. Also some special cases are discussed: the adjustment of row sums and column sums, additional restraints, mutual connections between tables and three dimensional tables.

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